Activities of the HEP-TH group

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HOCTools-II

INPP annual meeting 2024
March 28, & 29, 2024

March 28, 2024

Outline

1. Amplitude reduction
2. Master Integrals

NLO

Don't make integrals, make integrands !

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \dots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \quad \leftarrow LO \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \quad \leftarrow Virtual \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi) \quad \leftarrow Real
\end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safeness $J_{m+1} \to J_m$

PERTURBATIVE QCD AT NLO

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, ..., p_{m+2}$$

$$\begin{aligned}\sigma_{NLO} &= \int_m d\Phi_m^{D=4}(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV})))J_m(\Phi) \\ &+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR}))J_m(\Phi) \\ &+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)\end{aligned}$$

IR and UV divergencies, HV or Four-Dimensional-Helicity scheme; scale dependence μ_R

Perturbative QCD at NLO

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}\sigma_{NLO} &= \int_m d\Phi_m J_m(\Phi) \\ &+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\ &+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)\end{aligned}$$

QCD factorization$-\mu_F$ Collinear counter-terms when PDF are involved

The one loop paradigm

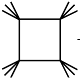
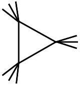
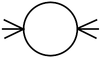
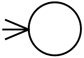
basis of scalar integrals: known already before NLO-R; remember this is not the case for higher orders

→G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.

→ Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412** (1994) 751

→G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

→Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217.

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad R \rightarrow$ rational terms

$$\mathcal{A} \quad = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

The old "master" formula

$$
\begin{aligned}
\mathcal{A} \to \int \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP R_1

$$
\begin{aligned}
\int d^n\bar{q}\,\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} &= -\frac{i\pi^2}{6}+\mathcal{O}(\epsilon)\,,\\
\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} &= -\frac{i\pi^2}{2}+\mathcal{O}(\epsilon)\,,\\
\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} &= -\frac{i\pi^2}{2}\left[m_i^2+m_j^2-\frac{(p_i-p_j)^2}{3}\right]+\mathcal{O}(\epsilon)\,.
\end{aligned}
$$

OPP R_1

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2\,, \quad p_0 \neq 0\,,$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$m_i^2 \to m_i^2 - \tilde{q}^2.$$

$$\begin{aligned} d(ijkl;\tilde{q}^2) &= d(ijkl) + \tilde{q}^2 d^{(2)}(ijkl) + \tilde{q}^4 d^{(4)}(ijkl)\,, \\ c(ijk;\tilde{q}^2) &= c(ijk) + \tilde{q}^2 c^{(2)}(ijk)\,, \\ b(ij;\tilde{q}^2) &= b(ij) + \tilde{q}^2 b^{(2)}(ij)\,. \end{aligned}$$

OPP R_1

$$
\begin{aligned}
d^{(4)}(ijkl) &= \lim_{\tilde{q}^2 \to \infty} \frac{d(ijkl;\tilde{q}^2)}{\tilde{q}^4}\,,\\
c^{(2)}(ijk) &= \lim_{\tilde{q}^2 \to \infty} \frac{c(ijk;\tilde{q}^2)}{\tilde{q}^2}\,,\\
b^{(2)}(ij) &= \lim_{\tilde{q}^2 \to \infty} \frac{b(ij;\tilde{q}^2)}{\tilde{q}^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
d^{(4)}(ijkl) &= \frac{d(ijkl;1) + d(ijkl;-1) - 2d(ijkl)}{2}\,,\\
c^{(2)}(ijk) &= c(ijk;1) - c(ijk)\,,\\
b^{(2)}(ij) &= b(ij;1) - b(ij)\,.
\end{aligned}
$$

OPP R_1

$$\mathrm{R}_1 = -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2)$$
$$- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) .$$

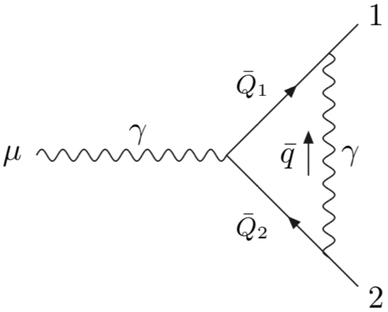
OPP R_2

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, q, \epsilon)\,.$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_{\mu} + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4}\int d^n\,\bar{q}\,\frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4}\int d^n\,\bar{q}\,\mathcal{R}_2\,.$$

OPP R_2



$$\bar{Q}_1 = \bar{q} + p_1 = Q_1 + \tilde{q}$$
$$\bar{Q}_2 = \bar{q} + p_2 = Q_2 + \tilde{q}$$

$$\bar{D}_0 = \bar{q}^2$$
$$\bar{D}_1 = (\bar{q} + p_1)^2$$
$$\bar{D}_2 = (\bar{q} + p_2)^2$$

Figure 1: QED $\gamma e^+ e^-$ diagram in n dimensions.

OPP R_2

ϵ-dimensional γ matrices freely anti-commute with four-dimensional ones:
$\{\gamma_\mu, \tilde{\gamma}_\nu\} \; = 0$

$$
\begin{aligned}
\bar{N}(\bar{q}) &\equiv e^3 \left\{ \bar{\gamma}_{\bar{\beta}} \, (\bar{\not{Q}}_1 + m_e) \, \gamma_\mu \, (\bar{\not{Q}}_2 + m_e) \, \bar{\gamma}^{\bar{\beta}} \right\} \\
&= e^3 \, \{ \gamma_\beta (\not{Q}_1 + m_e) \gamma_\mu (\not{Q}_2 + m_e) \gamma^\beta \\
&- \epsilon \, (\not{Q}_1 - m_e) \gamma_\mu (\not{Q}_2 - m_e) + \epsilon \tilde{q}^2 \, \gamma_\mu - \tilde{q}^2 \, \gamma_\beta \gamma_\mu \gamma^\beta \} \; ,
\end{aligned}
$$

$$
\begin{aligned}
\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_0 \bar{D}_1 \bar{D}_2} &= -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon) \, , \\
\int d^n \bar{q} \frac{q_\mu q_\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2} &= -\frac{i\pi^2}{2\epsilon} g_{\mu\nu} + \mathcal{O}(1) \, ,
\end{aligned}
$$

gives

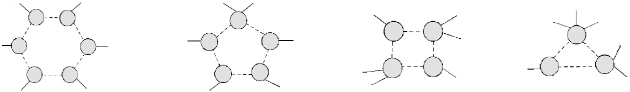
$$
\mathrm{R}_2 = -\frac{ie^3}{8\pi^2} \gamma_\mu + \mathcal{O}(\epsilon) \, ,
$$

1. Determining the on-shell momenta through $D_i = 0$ and computing all coefficients.
2. Determining the on-shell momenta through $D_i = \mu$ and μ dependence of certain coefficients, namely R_1.
3. Using new Feynman rules to compute the rest of R contribution, namely R_2.

The one-loop calculation in a nutshell

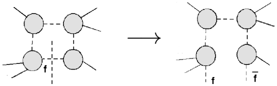
The computation of $pp(p\bar{p}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, `HELAC` evaluates numerically the numerators $N_i^6(q), N_i^5(q), \ldots$ with the values of the loop momentum q provided by `CutTools` $+R_1$.

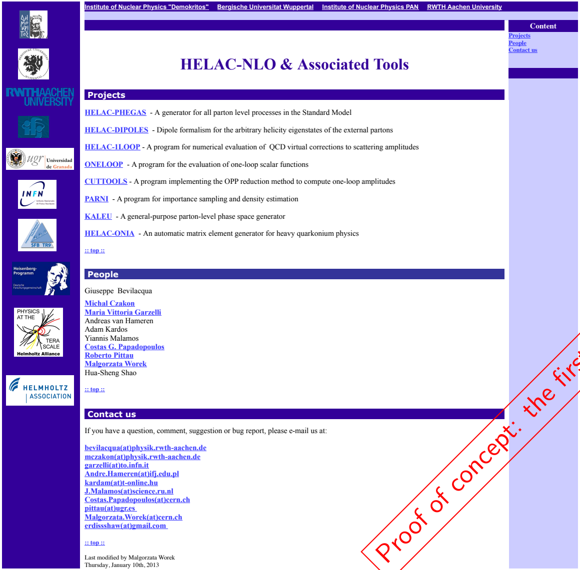
- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

$\to$ `BlackHat`, `MadGraph`, `RECOLA`, `OpenLoops`

The one-loop calculation in a nutshell


Institute of Nuclear Physics "Demokritos" Bergische Universitat Wuppertal Institute of Nuclear Physics PAN RWTH Aachen University

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HELAC-NLO & Associated Tools

Projects

HELAC-PHEGAS - A generator for all parton level processes in the Standard Model

HELAC-DIPOLES - Dipole formalism for the arbitrary helicity eigenstates of the external partons

HELAC-1LOOP - A program for numerical evaluation of QCD virtual corrections to scattering amplitudes

ONELOOP - A program for the evaluation of one-loop scalar functions

CUTTOOLS - A program implementing the OPP reduction method to compute one-loop amplitudes

PARNI - A program for importance sampling and density estimation

KALEU - A general-purpose parton-level phase space generator

HELAC-ONIA - An automatic matrix element generator for heavy quarkonium physics

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People

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Contact us

If you have a question, comment, suggestion or bug report, please e-mail us at:

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Last modified by Malgorzata Worek
Thursday, January 10th, 2013

Proof of concept: the first NLO public code

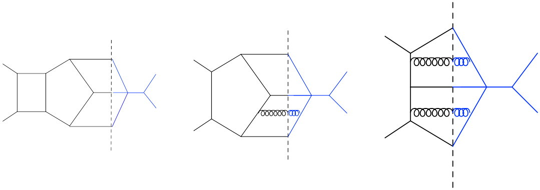
Towards higher precision:

NNLO and beyond

I have a dream ...

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$



What do we need for an NNLO calculation ?

$$
\begin{aligned}
\sigma_{NNLO} \quad \rightarrow \quad & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) & \quad VV \\
+ \quad & \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) & \quad RV \\
+ \quad & \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) & \quad RR
\end{aligned}
$$

$RV + RR \rightarrow$ antenna-S, colorfull-NNLO, sector-improved residue subtraction, nested soft-collinear, local analytic sector subtraction, projection to born, q_T, N-jetiness

→A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
→ P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
→ M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
→S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
→ R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002
→ M. Cacciari, F. A. Dreyer, A. Karlberg, G. P. Salam and G. Zanderighi, Phys. Rev. Lett. **115**, no. 8, 082002 (2015)
→ F. Caola, K. Melnikov and R. Röntsch, Eur. Phys. J. C **77**, no. 4, 248 (2017)
→ L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, arXiv:1806.09570 [hep-ph].

OPP@2L WITH HELAC

- Two-loop skeleton
  $\rightarrow$ D. Canko last year's talk and PhD thesis
- Amplitude reduction
- Master Integrals

OPP@2L WITH HELAC

- Amplitude reduction in 4 dimensions

- Cut equations
- Integrand basis
- R_1 terms
- R_2 terms

OPP@2L with HELAC

- Amplitude reduction in dim-reg.

$$\bar{q} = q + \tilde{q} \quad \bar{\gamma}^{\mu} = \gamma^{\mu} + \tilde{\gamma}^{\mu} \quad \bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}$$

$$\mu = \tilde{q} \cdot \tilde{q} = \tilde{\not{q}}\tilde{\not{q}}$$

$$d - 4 = \tilde{g}^{\mu\nu}\tilde{g}_{\mu\nu} = \tilde{\gamma}^{\mu}\tilde{\gamma}_{\mu}$$

- Amplitude evaluation

$$N(q) \quad or \quad N(\bar{q}, d) = N(q, \mu, d)$$

OPP@2L WITH HELAC

- Giuseppe Bevilacqua

How to compute

$$N(q, \mu, d)$$

- Aris Spourdalakis

How to reduce

$$N = P_{max-cut} + \sum_i P_{n-to-max-cut} D_i + \sum_{ij} P_{n-n-to-max-cut} D_i D_j + \ldots$$

where all the P are polynomials in the so-called irreducible and transverse scalar products.

- What do we expect at the end?

$$\mathcal{A} = \sum_i c_i F_i$$

c_i depends on the external world
F_i are Feynman integrals of the form

$$F_i \equiv F_{a_1 \ldots a_N} = \int d^d k \frac{\overbrace{(D_{m+1})^{a_{m+1}} \cdots (D_N)^{a_N}}^{ISP}}{\underbrace{(D_1)^{a_1} \cdots (D_m)^{a_m}}_{RSP}}$$

$$a_1, \ldots, a_m \rightarrow 1 \ (2) \quad a_{m+1}, \ldots, a_N \rightarrow \textit{number of external legs}$$

that through IBP tables will be expressed in terms of Master Integrals.
$\rightarrow$ full numerical evaluation of pole and finite-remainder terms

The two-loop frontier: $2 \to 3$ MI

5-POINT 2-LOOP - MASSLESS: ALL FAMILIES

→ T. Gehrmann, J. M. Henn and N. A. Lo Presti, Phys. Rev. Lett. **116** (2016) no.6, 062001 [erratum: Phys. Rev. Lett. **116** (2016) no.18, 189903] [arXiv:1511.05409 [hep-ph]].

→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **04** (2016), 078 [arXiv:1511.09404 [hep-ph]].

→ D. Chicherin, T. Gehrmann, J. M. Henn, P. Wasser, Y. Zhang and S. Zoia, Phys. Rev. Lett. **123** (2019) no.4, 041603

→ D. Chicherin and V. Sotnikov, JHEP **20** (2020), 167

→ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, "Caravel: A C++ framework for the computation of multi-loop amplitudes with numerical unitarity," Comput. Phys. Commun. **267** (2021), 108069

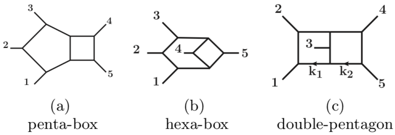
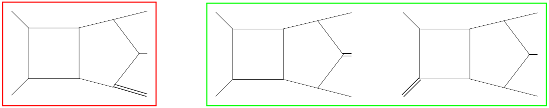


FIG. 1: Integral topologies for massless five-particle scattering at two loops.

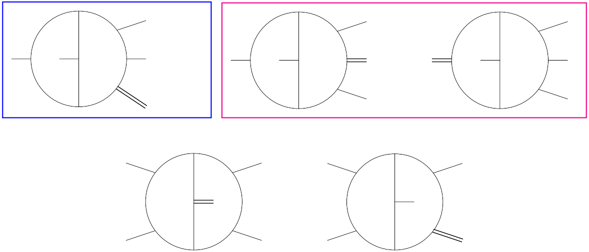
→ J. Henn, T. Peraro, Y. Xu and Y. Zhang, "A first look at the function space for planar two-loop six-particle Feynman integrals," JHEP **03** (2022), 056

5-POINT 2-LOOP - ONE LEG OFF-SHELL: ALL FAMILIES

→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **04** (2016), 078 [arXiv:1511.09404 [hep-ph]].
→C. G. Papadopoulos and C. Wever, JHEP **2002** (2020) 112
→S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117
→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199
→ S. Abreu, H. Ita, B. Page and W. Tschernow, JHEP **03** (2022), 182 [arXiv:2107.14180 [hep-ph]].
→ A. Kardos, C. G. Papadopoulos, A. V. Smirnov, N. Syrrakos and C. Wever, [arXiv:2201.07509 [hep-ph]].
→ S. Abreu, D. Chicherin, H. Ita, B. Page, V. Sotnikov, W. Tschernow and S. Zoia, [arXiv:2306.15431 [hep-ph]].

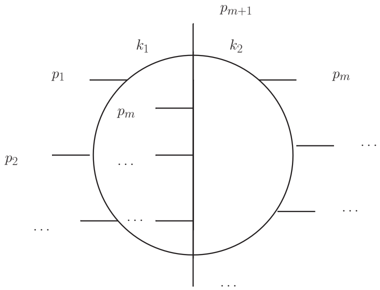


The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.



The five non-planar families with one external massive leg.

Feynman Integrals



$$\frac{\mathcal{N}\left(k_1, k_2; \{p_i\}_{i=1}^{m+1}, \{\varepsilon\}\right)}{\left(k^2 - {M_0}^2\right)\left((k_1+p_1)^2 - {M_1}^2\right)\ldots\left((k_1-k_2-p_{m+1})^2 - {M_j}^2\right)\ldots\left({k_2}^2 - {M_l}^2\right)}$$

The current approach

- m independent momenta, L loops, $N = L(L+1)/2 + Lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k_1, k_2\} + p_i)^2 - M_i^2$
- Definition

$$F[a_1, \dots, a_N] = C_L \int \frac{1}{D_1^{a_1} \dots D_N^{a_N}} \prod_{i=1}^{L} \left[d^d k_i \right]$$

with a_i being zero, positive or negative integers.

- Feynman parameters, Mellin-Barnes, Differential Equations
- Or numerical: `SecDec`, Weinzierl, Anastasiou&Sterman

→ C. Anastasiou and G. Sterman, arXiv:1812.03753 [hep-ph].

→ S. Kromin, N. Schwanemann and S. Weinzierl, [arXiv:2208.01060 [hep-th]].

The current approach

- m independent momenta, L loops, $N = L(L+1)/2 + Lm$ scalar products
- basis composed by $D_1 \ldots D_N$, allows to express all scalar products $D_i = (\{k_1, k_2\} + p_i)^2 - M_i^2$
- Definition
$$F[a_1, \ldots, a_N] = C_L \int \frac{1}{D_1^{a_1} \ldots D_N^{a_N}} \prod_{i=1}^{L} \left[d^d k_i\right]$$
with a_i being zero, positive or negative integers.
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→F. V. Tkachov, Phys. Lett. B **100** (1981) 65.

→K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192** (1981) 159.

IBP identities:

$$\int d^d k d^d l \;\; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, \upsilon^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$

reduce *all* Feynman Integrals to a finite subset → Master Integrals.

$$F[a_1, \dots, a_N] = \sum_i R_i\left(\{p\}, d\right) G_i\left[a'_1, \dots, a'_N\right]$$

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→ Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

→ V. A. Smirnov, Phys. Lett. B **460** (1999) 397

→ T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

→ J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

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→S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, Comput. Phys. Commun. **196** (2015) 470

→ S. Becker, C. Reuschle and S. Weinzierl, JHEP **1012** (2010) 013

→ C. Anastasiou and G. Sterman, arXiv:1812.03753 [hep-ph].

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Differential Equations Approach

- The integral is a function of external momenta, so one can set-up differential equations by differentiating and using IBP

$$F[a_1, \ldots, a_N] \to G[a'_1, \ldots, a'_N]$$

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum C_{b_1,\ldots,b_n} F[b_1, \ldots, b_n] \to \sum C_{a'_1,\ldots,a'_n} G[a'_1, \ldots, a'_n]$$

- Find the proper basis; Bring the system of equations in a form suitable to express the MI in terms of GPs

$$\partial_m f(\varepsilon, \{x_i\}) = \varepsilon A_m(\{x_i\}) f(\varepsilon, \{x_i\})$$
$$\partial_m A_n - \partial_n A_m = 0 \quad [A_m, A_n] = 0$$

$\star$ f not MI!

→ J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

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Differential Equations Approach

- Iterated Integrals →K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Multiple Polylogarithms, Symbol algebra
- Goncharov Polylogarithms

$$\mathcal{G}(a_n, \ldots, a_1, x) = \int_0^x dt \frac{1}{t - a_n} \mathcal{G}(a_{n-1}, \ldots, a_1, t)$$

→ J. Vollinga and S. Weinzierl, Comput. Phys. Commun. **167** (2005), 177

- Elliptic Integrals

→ L. Adams and S. Weinzierl, Phys. Lett. B **781** (2018), 270-278

→ J. Broedel, C. Duhr, F. Dulat, B. Penante and L. Tancredi, JHEP **01** (2019), 023

- Numerical approach [one-mass double-pentagon]
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→ X. Liu and Y. Q. Ma, Comput. Phys. Commun. **283** (2023), 108565

Differential Equations Approach

- Iterated Integrals
- Multiple Polylogarithms, Symbol algebra

→ A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

→C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

→C. Bogner and F. Brown

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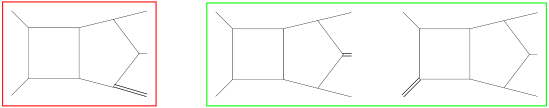
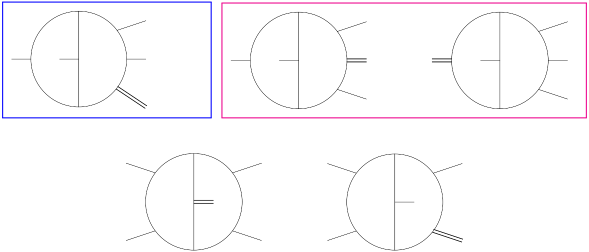
The SDE approach

→ C. G. Papadopoulos, "Simplified differential equations approach for Master Integrals," JHEP **1407** (2014) 088 [arXiv:1401.6057 [hep-ph]].



The three planar pentaboxes of the families P_1 (left), P_2 (middle) and P_3 (right) with one external massive leg.



The five non-planar families with one external massive leg.

Pentabox - one leg off-shell: P1

→J. M. Henn, Phys. Rev. Lett. **110** (2013) 251601

→S. Abreu, H. Ita, F. Moriello, B. Page, W. Tschernow and M. Zeng, JHEP **2011** (2020) 117

→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, JHEP **2101** (2021) 199

$$d\vec{g} = \epsilon \sum_a d\log\left(W_a\right) \tilde{M}_a \vec{g}$$

- Also from direct differentiation of MI wrt to x (Fuchsian).

$$\frac{d\vec{g}}{dx} = \epsilon \sum_b \frac{1}{x - \ell_b} M_b \vec{g}$$

- ℓ_b, are independent of x, some depending only on the reduced invariants, $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}\}$. M_b are independent of the invariants.
- number of letters smaller than in AIMPTZ representation
- Main contribution for us from AIMPTZ: the canonical basis (+ numerics)

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$$\frac{d\log\left(W_a\right)}{dx}$$

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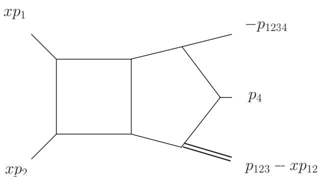


$$q_1 \to p_{123} - xp_{12}, \; q_2 \to p_4, \; q_3 \to -p_{1234}, \; q_4 \to xp_1$$

SDE parametrisation: n off-shell legs $\to$ $n-1$ off-shell legs + the x variable.

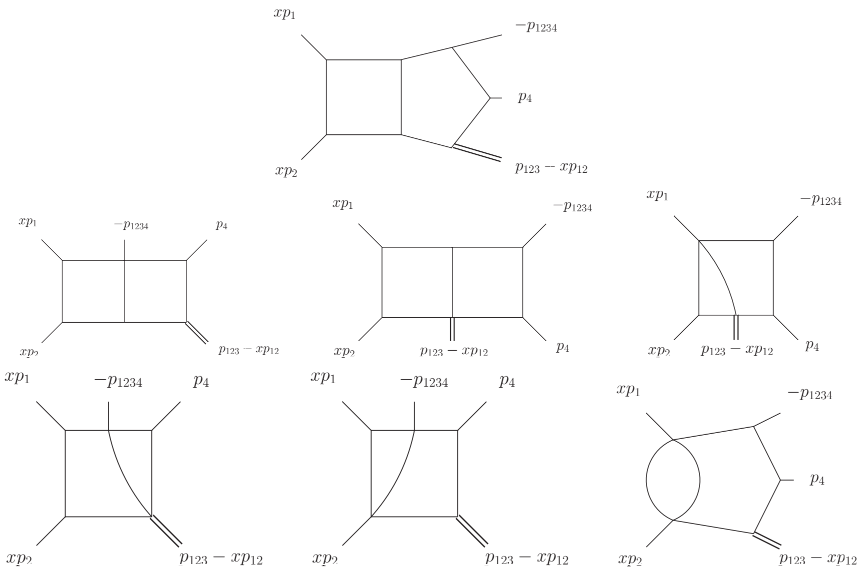
→ C. G. Papadopoulos, "Simplified differential equations approach for Master Integrals," JHEP **1407** (2014) 088

- $p_i, \; i = 1 \ldots 5$, satisfy $\sum_1^5 p_i = 0$, with $p_i^2 = 0, \; i = 1 \ldots 5, \; p_{i \ldots j} := p_i + \ldots + p_j$. The set of independent invariants: $\{S_{12}, S_{23}, S_{34}, S_{45}, S_{51}, x\}$, with $S_{ij} := (p_i + p_j)^2$.

$$q_1^2 = (1-x)(S_{45} - S_{12}x), \; s_{12} = (S_{34} - S_{12}(1-x))\,x, \; s_{23} = S_{45}, \; s_{34} = S_{51}x,$$

$$s_{45} = S_{12}x^2, \; s_{15} = S_{45} + (S_{23} - S_{45})x$$

Pentabox - one leg off-shell: P1



→ J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1405** (2014) 090
→ T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs, JHEP **06** (2014), 032
→ F. Caola, J. M. Henn, K. Melnikov and V. A. Smirnov, JHEP **1409** (2014) 043
→ C. G. Papadopoulos, D. Tommasini and C. Wever, JHEP **1501** (2015) 072
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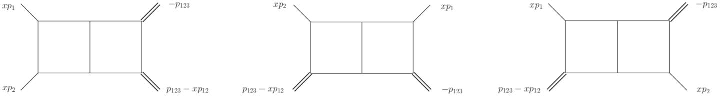

Figure 3. The parametrization of external momenta for the three planar double boxes of the families P_{12} (left), P_{13} (middle) and P_{23} (right) contributing to pair production at the LHC. All external momenta are incoming.

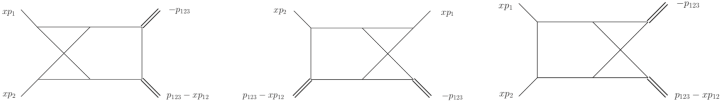

Figure 4. The parametrization of external momenta for the three non-planar double boxes of the families N_{12} (left), N_{13} (middle) and N_{34} (right) contributing to pair production at the LHC. All external momenta are incoming.

As well as planar and nonplanar double box with one off-shell leg expressed in UT basis.

PENTABOX - ONE LEG OFF-SHELL: P1-3

$$\frac{d\mathbf{g}}{dx} = \epsilon \sum_a \frac{1}{x - \ell_a} \mathbf{M}_a \mathbf{g}$$

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$$\frac{d\mathbf{g}}{dx} = \epsilon \sum_a \frac{1}{x - \ell_a} \mathbf{M}_a \mathbf{g}$$

$$\begin{aligned}
\mathbf{g} = {} & \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon \left(\sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) \\
& + \epsilon^2 \left(\sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(0)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) \\
& + \epsilon^3 \left(\sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(1)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(2)} + \mathbf{b}_0^{(3)} \right) \\
& + \epsilon^4 \left(\sum \mathcal{G}_{abcd} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{M}_d \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{abc} \mathbf{M}_a \mathbf{M}_b \mathbf{M}_c \mathbf{b}_0^{(1)} \right. \\
& \left. + \sum \mathcal{G}_{ab} \mathbf{M}_a \mathbf{M}_b \mathbf{b}_0^{(2)} + \sum \mathcal{G}_a \mathbf{M}_a \mathbf{b}_0^{(3)} + \mathbf{b}_0^{(4)} \right) + \ldots
\end{aligned}$$

$$\mathcal{G}_{ab\ldots} := \mathcal{G}(\ell_a, \ell_b, \ldots; x)$$

→ D. D. Canko, C. G. Papadopoulos and N. Syrrakos, arXiv:2009.13917 [hep-ph]. Results.txt

Pentabox - one leg off-shell: Kinematical Regions

- Euclidean region:

$$\left\{ \mathrm{S12} \to -2, \mathrm{S23} \to -3, \mathrm{S34} \to -5, \mathrm{S45} \to -7, \mathrm{S51} \to -11, x \to \frac{1}{4} \right\}$$

no letter l in the region $[0, x]$, all boundary terms real. [very fast GiNaC]

Family	W=1	W=2	W=3	W=4
P_1 (g_{72})	17 (14)	116 (95)	690 (551)	2740 (2066)
P_2 (g_{73})	25 (14)	170 (140)	1330 (1061)	4950 (3734)
P_3 (g_{84})	22 (12)	132 (90)	1196 (692)	4566 (2488)

TABLE: Number of GP entering in the solution. In parenthesis we give the corresponding number for the non-zero top-sector basis elements.

- with timings, running the GiNaC Interactive Shell `ginsh`, given by 1.9, 3.3, and 2 seconds for P_1, P_2 and P_3 respectively and for a precision of 32 significant digits
- A very different canonical basis, several elements start at ϵ^4.

Pentabox - one leg off-shell: Kinematical Regions

- One-scale integrals - closed form

$$
\begin{aligned}
(-s_{34})^{-\epsilon} &= (-S_{51})^{-\epsilon} x^{-\epsilon} \\
(-s_{45})^{-\epsilon} &= (-S_{12})^{-\epsilon} x^{-2\epsilon} \\
(-s_{15})^{-\epsilon} &= (-S_{45})^{-\epsilon} \left(1 - \frac{S_{45} - S_{23}}{S_{45}} x\right)^{-\epsilon} \\
(-p_{1s})^{-\epsilon} &= (1-x)^{-\epsilon} (-S_{45})^{-\epsilon} \left(1 - \frac{S_{12}}{S_{45}} x\right)^{-\epsilon} \\
(-s_{12})^{-\epsilon} &= x^{-\epsilon} (S_{12} - S_{34})^{-\epsilon} \left(1 - \frac{S_{12}}{S_{12} - S_{34}} x\right)^{-\epsilon},
\end{aligned}
$$

- One-scale integrals - expanded form

$$
\begin{aligned}
&\mathrm{Log}[-\mathrm{p1s} - i\delta] \to G[1, x] + G\left[\frac{\mathrm{S45}}{\mathrm{S12}}, x\right] + \mathrm{Log}[-\mathrm{S45}], \\
&\mathrm{Log}[-\mathrm{s34} - i\delta] \to \mathrm{Log}[-\mathrm{S51}] + \mathrm{Log}[x], \\
&\mathrm{Log}[-\mathrm{s12} - i\delta] \to G\left[\frac{\mathrm{S12} - \mathrm{S34}}{\mathrm{S12}}, x\right] + \mathrm{Log}[\mathrm{S12} - \mathrm{S34}] + \mathrm{Log}[x], \\
&\mathrm{Log}[-\mathrm{s45} - i\delta] \to \mathrm{Log}[-\mathrm{S12}] + 2\,\mathrm{Log}[x], \\
&\mathrm{Log}[-\mathrm{s15} - i\delta] \to G\left[\frac{\mathrm{S45}}{-\mathrm{S23} + \mathrm{S45}}, x\right] + \mathrm{Log}[-\mathrm{S45}]
\end{aligned}
$$

Pentabox - one leg off-shell: Kinematical Regions

- In general many letters will be now in $[0, x]$. This has two consequences:
  1. Need to fix infinitesimal imaginary part of $\frac{l_i}{x}$
  2. Increasing CPU time in `GiNaC`.
- Since the $\mathcal{F}$ Symanzik-polynomial maintains the sign of the $i0$ prescription of Feynman propagators with all original invariants assuming $s_{ij}(p_{1s}) \to s_{ij}(p_{1s}) + i\delta$, we determine the corresponding infinitesimal imaginary part of $\frac{l_i}{x}$ from

$$p_{1s} + i\delta = (1-x)(S_{45} - S_{12}x),\; s_{12} + i\delta = (S_{34} - S_{12}(1-x))\,x,$$
$$s_{23} + i\delta = S_{45},\; s_{34} + i\delta = S_{51}x,$$
$$s_{45} + i\delta = S_{12}x^2,\; s_{15} + i\delta = S_{45} + (S_{23} - S_{45})x$$

  with $S_{ij} \to S_{ij} + i\delta\eta_{ij}$, $x \to x + i\delta\eta_x$,
- Building a Fibration Basis using for instance `PolyLogTools`.

→D. Chicherin, V. Sotnikov and S. Zoia, JHEP **01** (2022), 096

Pentabox - one leg off-shell: Validation

- All regions of AIMPTZ checked @precision

- One-loop pentagon at order $\mathcal{O}(\varepsilon^4)$ [any order, analytic]

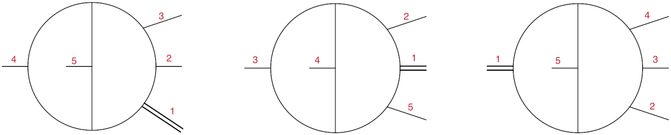
  → N. Syrrakos, "Pentagon integrals to arbitrary order in the dimensional regulator," arXiv:2012.10635 [hep-ph].

- Taken the limit $x = 1$ in all families to obtain the result for on-shell planar 5box

> SDE is not only capable to produce analytic results for off-shell MI but it can also give, almost for free, the on-shell MI.

- Evaluating phase-space points for $pp \to W^+ j_1 j_2$ generated by `HELAC-PHEGAS`, i.e. arbitrary floating points.

HEXABOX - ONE LEG OFF-SHELL



$$r_1 = \sqrt{\lambda(p_{1s},\ s_{23},\ s_{45})}$$

$$r_2 = \sqrt{\lambda(p_{1s},\ s_{24},\ s_{35})}$$

$$r_3 = \sqrt{\lambda(p_{1s},\ s_{25},\ s_{34})}$$

$$r_4 = \sqrt{\det \mathbb{G}(q_1,\ q_2,\ q_3,\ q_4)}$$

$$r_5 = \sqrt{\Sigma_5^{(1)}}$$

$$r_6 = \sqrt{\Sigma_5^{(2)}}$$

HEXABOX - ONE LEG OFF-SHELL

- For topology N_1, the square roots r_1 and r_4 appear in its alphabet and are rationalized.

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{l_{max}} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g}$$

  $l_{max} = 21$ from 39 letters in the original alphabet

- For topologies N_2 and N_3, the square roots appearing are $\{r_1,\ r_2,\ r_4,\ r_5\}$ and $\{r_1,\ r_3,\ r_4,\ r_6\}$ not *simultaneous* rationalisation possible !

The more general form of the SDE takes the form:

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{a=1}^{l_{max}} \frac{d \log L_a}{dx} \mathbf{M}_a \right) \mathbf{g}$$

where most of the L_a are simple rational functions of x , whereas the rest are algebraic functions of x involving the non-rationalisable square roots.

- One-dimensional integration based on weight-2 functions

→ S. Caron-Huot and J. M. Henn, JHEP **06** (2014), 114

For instance element 11 of N_2 is given as

$$g_{11}^{(2)} = 8\left(2\mathcal{G}(0,-y)\left(\mathcal{G}(1,y) - \mathcal{G}\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right)\right) + 2\mathcal{G}\left(0,\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right) - \mathcal{G}(1,y)\log\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}}\right) + \log\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}}\right)\mathcal{G}\left(\frac{\tilde{S}_{45}}{\tilde{S}_{12}},y\right) - 2\mathcal{G}(0,1,y)\right)$$

where the new parametrization of the external momenta is given by

$$q_1 \to \tilde{p}_{123} - y\tilde{p}_{12},\; q_2 \to y\tilde{p}_2,\; q_3 \to -\tilde{p}_{1234},\; q_4 \to y\tilde{p}_1$$

with the new momenta $\tilde{p}_i,\; i = 1\ldots5$ satisfying as usual, $\sum_1^5 \tilde{p}_i = 0,\; \tilde{p}_i^2 = 0,\; i = 1\ldots5$, with $\tilde{p}_{i\ldots j} := \tilde{p}_i + \ldots + \tilde{p}_j$. The set of independent invariants is given by $\{\tilde{S}_{12}, \tilde{S}_{23}, \tilde{S}_{34}, \tilde{S}_{45}, \tilde{S}_{51}, y\}$, with $\tilde{S}_{ij} := (\tilde{p}_i + \tilde{p}_j)^2$. The explicit mapping between the two sets of invariants is given by

$$q_1^2 = (1-y)(\tilde{S}_{45} - \tilde{S}_{12}y),\; s_{12} = \tilde{S}_{45}(1-y) + \tilde{S}_{23}y,\; s_{23} = -y\left(\tilde{S}_{12} - \tilde{S}_{34} + \tilde{S}_{51}\right),$$

$$s_{34} = \tilde{S}_{51}y, s_{45} = y\left(\tilde{S}_{23} - \tilde{S}_{45} - \tilde{S}_{51}\right),\; s_{15} = y\left(\tilde{S}_{34} - \tilde{S}_{12}(1-y)\right).$$

HEXABOX - ONE LEG OFF-SHELL: WEIGHT 2

- By identifying $f_- = y$ and $f_+ = y\frac{\tilde{S}_{12}}{\tilde{S}_{45}}$, which are given as

$$f_\pm = \frac{S_{45} + x\left(-S_{23} - S_{34} + 2S_{51} + S_{12}x\right) \pm r_2}{2\left(S_{12} - S_{34} + S_{51}\right)x}$$

we can write the DE for this element in the simple and compact form

$$\frac{d}{dx}g_{11}^{(2)} = -8\left(\mathrm{dlog}\left(\frac{f_+ - 1}{f_- - 1}\right)\log\left(f_- f_+\right) - \mathrm{dlog}\left(\frac{f_+}{f_-}\right)\log\left(\left(f_- - 1\right)\left(f_+ - 1\right)\right)\right).$$

The form of the DE makes the determination of the ansatz rather straightforward, with the result

$$g_{11}^{(2)} = -8\Big(-\log(f_- f_+)\Big(\mathcal{G}(1, f_-) - \mathcal{G}(1, f_+)\Big) + 2\mathcal{G}(0, 1, f_-) - 2\mathcal{G}(0, 1, f_+)\Big).$$

- Concerning the other non-rationalisable square root in the family N_2, r_5, it also appears for the first time at weight 2 in the basis element 73 only , which is one of the new integrals to be calculated.

$$g_{73}^{(2)} = 16\log\left(f_- f_+\right)\left(\mathcal{G}(1, f_-) - \mathcal{G}(1, f_+)\right) - 32\left(\mathcal{G}(0, 1, f_-) - \mathcal{G}(0, 1, f_+)\right)$$

with

$$f_\pm = \frac{S_{45}\left(2S_{12}x - S_{34}x + S_{51}\right) + x\left(S_{23}S_{34} - S_{12}S_{23} + xS_{12}S_{51}\right) \pm r_5}{2S_{45}\left(S_{12} - S_{34} + S_{51}\right)}$$

HEXABOX - ONE LEG OFF-SHELL: INTEGRAL REP.

Weight 3:
The differential equation can be written in the form:

$$\partial_x g_I^{(3)} = \sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)}$$

Since the lower limit of integration corresponds to $x = 0$, we need to subtract the appropriate term so that the integral is explicitly finite. This is achieved as follows:

$$\partial_x g_I^{(3)} = \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} + \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right)$$

where $g_{I,0}^{(2)}$ are obtained by expanding $g_I^{(2)}$ around $x = 0$ and keeping terms up to order $\mathcal{O}\left(\log(x)^2\right)$, and $l_a \in \mathbb{Q}$ are defined through

$$\partial_x \log L_a = \frac{l_a}{x} + \mathcal{O}\left(x^0\right).$$

The DE can now be integrated from $x = 0$ to $x = \bar{x}$, and the result is given by

$$g_I^{(3)} = g_{I,G}^{(3)} + b_I^{(3)} + \int_0^{\bar{x}} dx \left(\sum_a (\partial_x \log L_a) \sum_J c_{IJ}^a g_J^{(2)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(2)} \right)$$

with $b_I^{(3)}$ being the boundary terms at $\mathcal{O}(\epsilon^3)$ and

$$g_{I,\mathcal{G}}^{(3)} = \int_0^{\bar{x}} \mathrm{d}x \sum_a \frac{l_a}{x} \sum_J c_{LJ}^a g_{J,0}^{(2)} \bigg|_{\mathcal{G}}$$

with the subscript $\mathcal{G}$, indicating that the integral is represented in terms of GPLs , following the convention

$$\int_0^{\bar{x}} dx \frac{1}{x} \mathcal{G}\left(\underbrace{0,\ldots 0}_{n}; x\right) = \mathcal{G}\left(\underbrace{0,\ldots 0}_{n+1}; \bar{x}\right) .$$

Alternative for the analytical aficionados (AA): work out *linear letters* →Goncharov MPL

→For instance, N_2 element 11 known at w=3 in terms of y, as well as many other

Weight 4:
At weight 4, the differential equation can be written in the form:

$$\partial_x g_I^{(4)} = \sum_a \Big(\partial_x \log L_a\Big) \sum_J c^a_{IJ} g_J^{(3)}$$

which after doubly-subtracting, in order to obtain integrals that are explicitly finite , is written as

$$\partial_x g_I^{(4)} = \sum_a \partial_x (\log L_a - LL_a) \sum_J c^a_{IJ} g_J^{(3)} + \sum_a \partial_x (LL_a) \sum_J c^a_{IJ} (g_J^{(3)} - g_{J,0}^{(3)}) + \sum_a \frac{l_a}{x} \sum_J c^a_{IJ} g_{J,0}^{(3)}$$

where LL_a are obtained by expanding $\log(L_a)$ around $x = 0$ and keeping terms up to order $\mathcal{O}\Big(\log(x)\Big)$, and

$$g_{I,0}^{(3)} = g_{I,\mathcal{G}}^{(3)} + b_I^{(3)}.$$

Now, by integrating by parts we can write the final result as follows:

$$
\begin{aligned}
g_I^{(4)} =& g_{I,\mathcal{G}}^{(4)} + b_I^{(4)} + \left(\sum_a \log L_a \sum_J c_{IJ}^a g_J^{(3)}\right) - \left(\sum_a LL_a \sum_J c_{IJ}^a g_{J,0}^{(3)}\right) \\
& - \int_0^{\bar{x}} \mathrm{d}x \sum_a (\log L_a - LL_a) \sum_J c_{IJ}^a \sum_b \frac{l_b}{x} \sum_K c_{JK}^b g_{K,0}^{(2)} \\
& - \int_0^{\bar{x}} \mathrm{d}x \sum_a \log L_a \sum_J c_{IJ}^a \left(\sum_b (\partial_x \log L_b) \sum_K c_{JK}^b g_K^{(2)} - \sum_b \frac{l_b}{x} \sum_K c_{JK}^b g_{K,0}^{(2)}\right)
\end{aligned}
$$

with a, b running over the set of contributing letters, I, J, K running over the set of basis elements, $b_I^{(4)}$ being the boundary terms at $\mathcal{O}(\epsilon^4)$ and

$$
g_{I,\mathcal{G}}^{(4)} = \int_0^{\bar{x}} \mathrm{d}x \left(\sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(3)}\right)\Bigg|_{\mathcal{G}}
$$

where the subscript $\mathcal{G}$ indicates that the integral is represented in terms of GPLs.

analytic continuation $\rightarrow$ applying fibration on $b_I^{(1\ldots4)}$ and g up to weight two

Generalising from $n \to n+1$ and from $n \to n+2$

$$g_I^{(n+1)} = g_{I,G}^{(n+1)} + b_I^{(n+1)} + \int_0^{\bar{x}} dx \, \left(\sum_a \left(\partial_x \log L_a\right) \sum_J c_{IJ}^a g_J^{(n)} - \sum_a \frac{l_a}{x} \sum_J c_{IJ}^a g_{J,0}^{(n)} \right)$$

Proper analytic continuation for

$$g_J^{(n\leq 2)}(x) \quad g_{J,0}^{(\leq 2)}$$

$$\partial_x \log L_a$$

$$\log L_a \quad LL_a$$

HEXABOX - ONE LEG OFF-SHELL: INTEGRAL REP.

$g^{(n\leq 2)}(x)$ depend on

$$L\left(x,\vec{S}\right)=\left\{\begin{array}{c} x-\ell\left(\vec{S}\right) \\ F\left(x,\vec{S}\right)\end{array}\right.$$

- For rational letters: Symanzik polynomial & one-scale MI
- For the non-rational letters: new approach

From $x=0$ to $x=\bar{x}$, crossing different regions (physical and non-physical)

Possible solutions:

→ X. Liu and Y. Q. Ma, Comput. Phys. Commun. **283** (2023), 108565

→ D. Chicherin and V. Sotnikov, JHEP **20** (2020), 167

→N. Dokmetzoglou

Summary & Outlook

- Amplitude reduction
  - Cut equations: determining on-shell loop momenta
  - Integrand basis construction and fitting
  - R_1 and R_2 terms, if needed
  - IBP reduction tables
  - 4d or dim-reg
- Master Integrals
  - Understanding the analytic continuation
  - Computing the last two non-planar double-pentagon families
  - Internal masses

→ N. Syrrakos, JHEP **10** (2021), 041 [arXiv:2107.02106 [hep-ph]].

→ S. Badger, M. Becchetti, E. Chaubey and R. Marzucca, JHEP **01** (2023), 156 [arXiv:2210.17477 [hep-ph]].

  - Using semi-numerical results

→ https://gitlab.com/pentagon-functions/PentagonFunctions-cpp

Thank you for your attention !

The research project was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the 2nd Call for H.F.R.I. Research Projects to support Faculty Members & Researchers (Project Number: 2674).

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