NATIONAL CENTRE FOR
SCIENTIFIC RESEARCH "DEMOKRITOS"
INSTITUTE OF NUCLEAR AND PARTICLE PHYSICS

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Towards numerical two-loop integrand reduction

Giuseppe Bevilacqua
NCSR "Demokritos"

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In collaboration with D. Canko, C. Papadopoulos and A. Spourdalakis

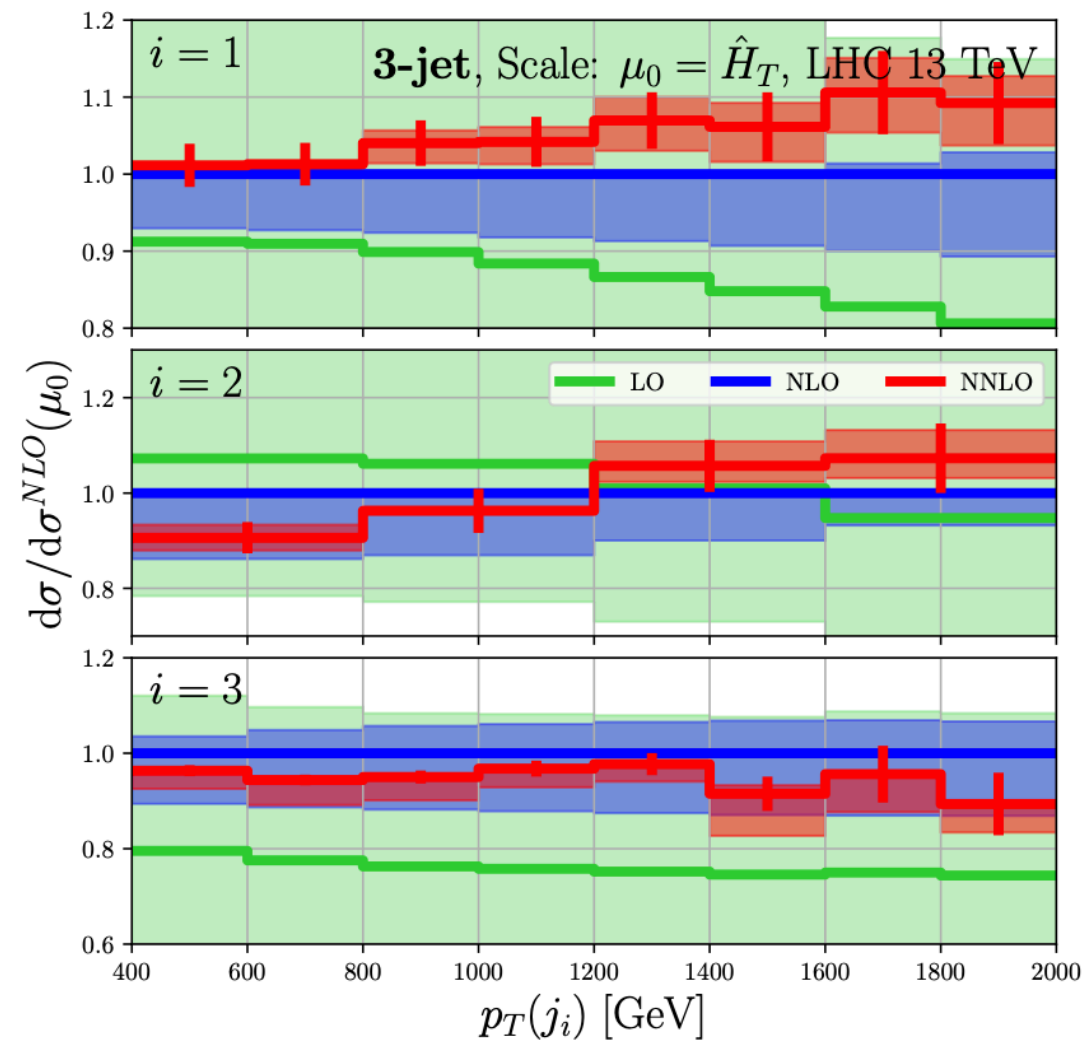
Based on arXiv:2506.07231 [hep-ph] + ongoing work

Introduction

- After the "NLO revolution" has set, NNLO is becoming the new theory standard for many QCD processes at the LHC

$$
\begin{aligned}
\sigma_{NNLO} = & \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left|M_m^{(1)}\right|^2\right) J_m(\Phi) && \text{[VV]} \\
& + \int_{m+1} d\Phi_{m+1} \left(2Re\left(M_{m+1}^{(0)*} M_{m+1}^{(1)}\right)\right) J_{m+1}(\Phi) && \text{[RV]} \\
& + \int_{m+2} d\Phi_{m+2} \left|M_{m+2}^{(0)}\right|^2 J_{m+2}(\Phi) && \text{[RR]}
\end{aligned}
$$

{Czakon, Mitov and Poncelet '21}



- The current NNLO frontier is $2 \to 3$ processes

[see also Les Houches 2023 report]

- $pp \to 3$ jets [Czakon, Mitov and Poncelet '21]
- $pp \to W^{\pm} b\bar{b}$ [Hartanto, Poncelet, Popescu and Zoia '22]
- $pp \to t\bar{t}H$ [Catani *et al.* '22; Devoto *et al.* '24]
- $pp \to t\bar{t}W$ [Buonocore *et al.* '23]
- $pp \to \gamma jj$ [Badger *et al.* '23]
- …

[selected examples]

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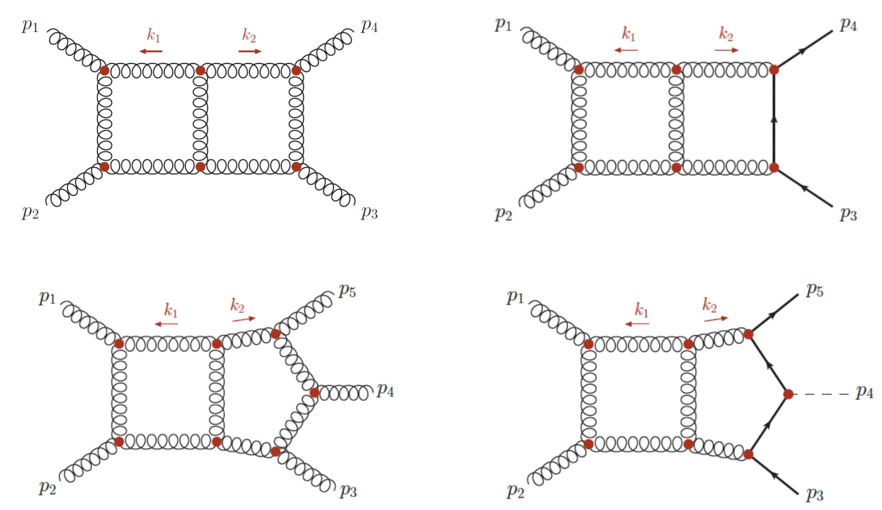
$$



- 2-loop amplitudes are crucial ingredients (and often bottlenecks) for NNLO calculations
  - *Construction* of 2-loop integrands — Feynman diagrams | Dyson-Schwinger recursion
    [Nogueira, Hahn, Ita, Pozzorini, Schar, Zoller, Papadopoulos, …]
  - *Reduction* of 2-loop integrals — Integrand reduction | IBP reduction + Finite Fields
    [Ita, Pozzorini, Schar, Zoller, Maierhofer, Lange, Usovitsch, Smirnov, von Manteuffel, Studerus, Peraro, Tancredi, Anastasiou, Mastrolia, Zhang, Badger, Chetyrkin, Tkachov, Laporta, …]
  - *Calculation* of MI's — Analytic | Numerical | Semi-numerical
    [Binoth, Heinrich, Gehrmann, Remiddi, Henn, Huang, …]

Introduction

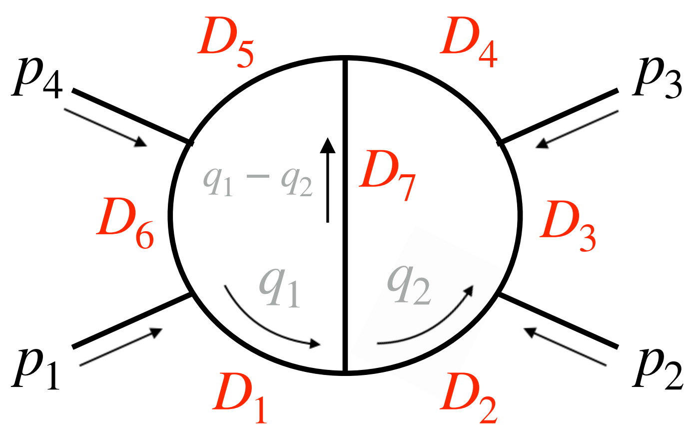
- So far, no complete framework for *automated numerical* 2-loop computations (as compared to the 1-loop case) is available*

  (*) ongoing efforts by HELAC and OpenLoops collaborations

- The path to 2-loop automation requires coordinated efforts in:
  - Extending *integrand-level reduction* to 2-loop calculations
  - Developing a numerical framework for the reduction method → "CutTools2"
  - Building 2-loop amplitude *integrands* → HELAC-2LOOP, OpenLoops
  - Evaluating Feynman integrals numerically (using e.g. IBP…)

In this talk we will discuss ongoing efforts in these directions, with emphasis on *integrand-level reduction*

- Example: 4-point function



$$A_{2L} = \int d^d\bar{q}_1\, d^d\bar{q}_2 \frac{N(\bar{q}_1, \bar{q}_2, \{p\})}{\prod_{i=1}^{7} D_i(\bar{q}_1, \bar{q}_2, \{p\})}$$

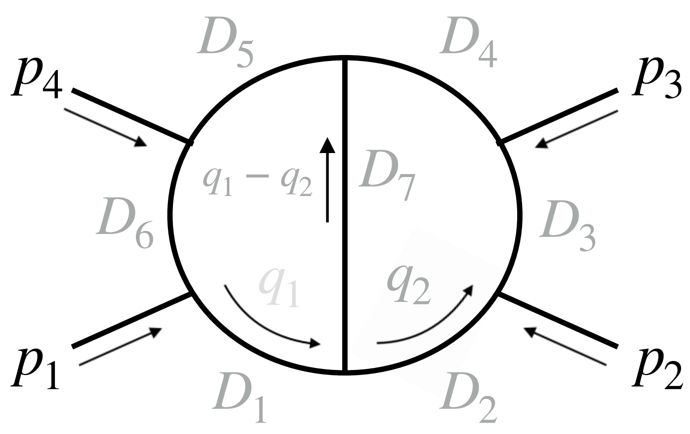
Denominator

“Inverse Propagators”

$$\begin{aligned}
D_1 &= \bar{q}_1^{\,2} \\
D_2 &= \bar{q}_2^{\,2} \\
D_3 &= (\bar{q}_2 + p_2)^2 \\
D_4 &= (\bar{q}_2 + p_2 + p_3)^2 \\
D_5 &= (\bar{q}_1 + p_2 + p_3)^2 \\
D_6 &= (\bar{q}_1 + p_2 + p_3 + p_4)^2 \\
D_7 &= (\bar{q}_1 - \bar{q}_2)^2
\end{aligned}$$

Anatomy of 2-loop amplitudes

- Example: 4-point function



Numerator

$$A_{2L} = \int d^d\bar{q}_1\, d^d\bar{q}_2 \, \frac{N(\bar{q}_1, \bar{q}_2, \{p\})}{\prod_{i=1}^{7} D_i(\bar{q}_1, \bar{q}_2, \{p\})}$$

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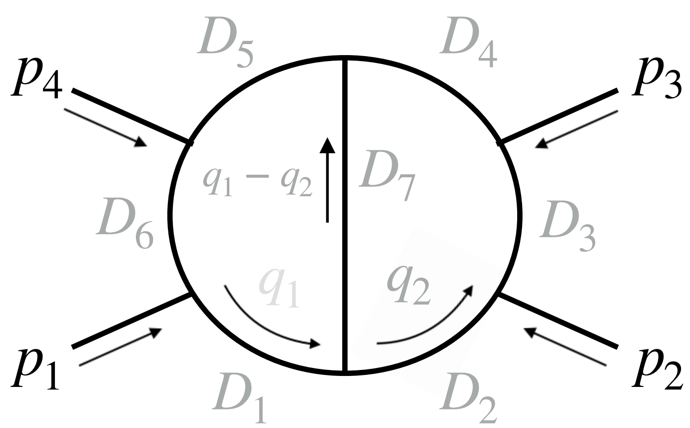
D_7 &= (\bar{q}_1 - \bar{q}_2)^2
\end{aligned}$$

The Numerator is a function of few constituents (Lorentz scalars):

$$(\bar{q}_i \cdot \bar{q}_j) \,,\, (\bar{q}_i \cdot p_j) \,,\, (p_i \cdot p_j) \,,\, (\bar{q}_i \cdot \eta)$$

$\longrightarrow (\eta \cdot p_j) \equiv 0$
[transverse vector]

- Example: 4-point function



$$A_{2L} = \int d^d\bar{q}_1\, d^d\bar{q}_2 \, \frac{N(\bar{q}_1, \bar{q}_2, \{p\})}{\prod_{i=1}^{7} D_i(\bar{q}_1, \bar{q}_2, \{p\})}$$

$$\begin{aligned} D_1 &= \bar{q}_1{}^2 \\ D_2 &= \bar{q}_2{}^2 \\ D_3 &= (\bar{q}_2 + p_2)^2 \\ D_4 &= (\bar{q}_2 + p_2 + p_3)^2 \\ D_5 &= (\bar{q}_1 + p_2 + p_3)^2 \\ D_6 &= (\bar{q}_1 + p_2 + p_3 + p_4)^2 \\ D_7 &= (\bar{q}_1 - \bar{q}_2)^2 \end{aligned}$$

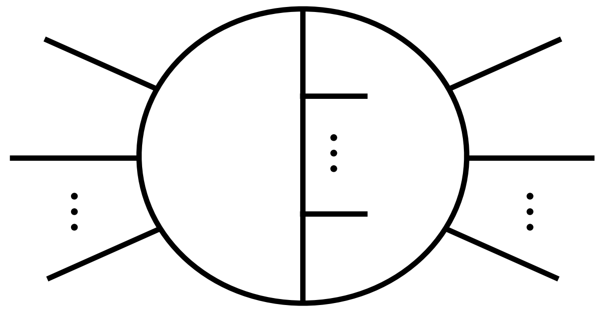
The Numerator is a function of few constituents (Lorentz scalars):

$$(\bar{q}_i \cdot \bar{q}_j)\ ,\ (\bar{q}_i \cdot p_j)\ ,\ (p_i \cdot p_j)\ ,\ (\bar{q}_i \cdot \eta)$$

- Some of the scalars above can be decomposed in terms of D_i's $\rightarrow$ *Reducible Scalar Products* (RSP)
- Other scalars do *not* allow such decomposition in terms of D_i's $\rightarrow$ *Irreducible Scalar Products* (ISP)

Aims and claims

- A generic 2-loop amplitude can be *reduced* into a set of simpler Feynman integrals:



$$\mathscr{A}_{2L} = \sum_i a_i F_i$$

- If we know how to:
  - determine the coefficients, → [Task 1]
  - decompose all Feynman integrals into *Master Integrals* (using IBP relations), → [Task 2]

  then we know how to compute $\mathscr{A}_{2L}$

- Goal: establish a *numerical* method which allows to perform the above tasks in *automated* form and for *arbitrary* processes (similarly to e.g. OPP method at 1-loop)

Focus of this talk → extracting coefficients at *integrand* level

Parametrising the Numerator

- The first step is to build a general parametrisation of the Numerator:

$$N(\bar{q}_1, \bar{q}_2, \{p\}) = P^{(m)} + \sum_i P_i^{(m-1)} D_i + \sum_{i<j} P_{ij}^{(m-2)} D_i D_j + \sum_{i<j<k} P_{ij}^{(m-3)} D_i D_j D_k + \cdots$$

$$A_{2L} = \int d^d\bar{q}_1\, d^d\bar{q}_2 \frac{N(\bar{q}_1, \bar{q}_2, \{p\})}{\prod_{i=1}^{7} D_i(\bar{q}_1, \bar{q}_2, \{p\})}$$

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- The $P^{(x)}$'s are *polynomials* built out of the available ISP's :

$$P^{(x)} = \sum_k c_k M_k(\bar{q}_1, \bar{q}_2, \{p\}) \quad \rightarrow \quad \text{e.g.: } M_k = (\bar{q}_1 \cdot p_2)(\bar{q}_2 \cdot p_4)^3$$

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$$c_k = c_k(\,\{p\}\,)$$

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- We need an *ansatz* to characterise the polynomials:
    - How many terms ?
    - Analytic structure of each term ? → using **`BasisDet`** [Zhang '12]
    - Maximal power ?

Fitting coefficients

$$N(\bar{q}_1, \bar{q}_2, \{p\}) = P^{(m)} + \sum_i P_i^{(m-1)} D_i + \sum_{i<j} P_{ij}^{(m-2)} D_i D_j + \sum_{i<j<k} P_{ij}^{(m-3)} D_i D_j D_k + \cdots$$

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Once the ansatz is established, fit the coefficients (c_k)

- Method n.1: "global fit" ⟷ Method n.2: "fit by cut"

$$N(\bar{q}_1, \bar{q}_2, \{p\}) = P^{(m)} + \sum_i P_i^{(m-1)} D_i + \sum_{i<j} P_{ij}^{(m-2)} D_i D_j + \sum_{i<j<k} P_{ij}^{(m-3)} D_i D_j D_k + \cdots$$

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Method n.1: "global fit"

- given the external kinematics ($\{p\}$), sample $N(\bar{q}_1, \bar{q}_2, \{p\})$ with *random* values of $\{\bar{q}_1, \bar{q}_2\}$

$$\underbrace{\begin{pmatrix} N_1 \\ \vdots \\ N_m \end{pmatrix} = \begin{pmatrix} M_{11} & \cdots & M_{1m} \\ & \ddots & \\ M_{m1} & \cdots & M_{mm} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}}_{\text{Linear system}}$$

- coefficients of all polynomials are extracted at the same time
- *huge* system of equations! Not the most efficient method, but feasible numerically

↪ `LAPACK, Eigen`

Fitting coefficients

$$N(\bar{q}_1, \bar{q}_2, \{p\}) = P^{(m)} + \sum_i P_i^{(m-1)} D_i + \sum_{i<j} P_{ij}^{(m-2)} D_i D_j + \sum_{i<j<k} P_{ij}^{(m-3)} D_i D_j D_k + \cdots$$

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$$P^{(x)} = \sum_k c_k M_k(\bar{q}_1, \bar{q}_2, \{p\})$$

Method n.2: "fit by cut"

- Sample N using sets of $\{\bar{q}_1, \bar{q}_2\}$ such that *all* D_k's vanish (→ "*maximal-cuts*"). Extract coefficients of $P^{(m)}$;

$$N(\bar{q}_1, \bar{q}_2, \{p\}) - P^{(m)} = \sum_i P_i^{(m-1)} D_i + \sum_{i<j} P_{ij}^{(m-2)} D_i D_j + \sum_{i<j<k} P_{ij}^{(m-3)} D_i D_j D_k + \cdots$$

$$P^{(x)} = \sum_k c_k M_k(\bar{q}_1, \bar{q}_2, \{p\})$$

Once the ansatz is established, fit the coefficients (c_k)

Method n.2: "fit by cut"

- Sample N using sets of $\{\bar{q}_1, \bar{q}_2\}$ such that *all* D_k's vanish ($\rightarrow$ "*maximal-cuts*"). Extract coefficients of $P^{(m)}$;
- Sample $N - P^{(m)}$ using sets of $\{\bar{q}_1, \bar{q}_2\}$ such that *all* D_k's *except* D_i vanish ($\rightarrow$ "*next-to-maximal cuts*"). Extract coefficients of $P_i^{(m-1)}$;

Fitting coefficients

$$N(\bar{q}_1, \bar{q}_2, \{p\}) - P^{(m)} - \sum_i P_i^{(m-1)} D_i = \sum_{i<j} P_{ij}^{(m-2)} D_i D_j + \sum_{i<j<k} P_{ij}^{(m-3)} D_i D_j D_k + \cdots$$

$$P^{(x)} = \sum_k c_k M_k(\bar{q}_1, \bar{q}_2, \{p\})$$

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⋮

- Proceed iteratively till all polynomials are reconstructed

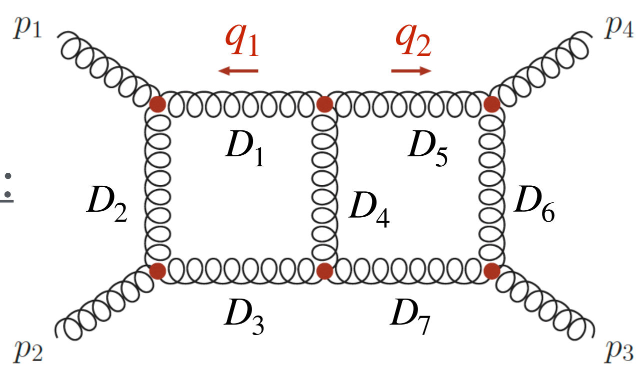
↪ Consistency check: *original* Numerator = *ansatz* Numerator ("$N = N$ test")

Constraints from cut equations

$$N(\bar{q}_1, \bar{q}_2, \{p\}) = P^{(m)} + \sum_{i=1}^{7} P_i^{(m-1)} D_i + \sum_{i<j}^{7} P_{ij}^{(m-2)} D_i D_j + \sum_{i<j<k}^{7} P_{ij}^{(m-3)} D_i D_j D_k + \cdots$$

- Cut equations constrain the ISP's appearing in the polynomials

e.g.:



$$D_1 = q_1^2, \quad D_2 = (q_1 + p_1)^2, \quad D_3 = (q_1 + p_{12})^2, \quad D_4 = (q_1 + q_2)^2, \quad D_5 = q_2^2,$$
$$D_6 = (q_2 - p_{123})^2, \quad D_7 = (q_2 - p_{12})^2$$

- Maximal cut equation: $D_1 = D_2 = D_3 = D_4 = D_5 = D_6 = D_7 = 0$ $\qquad s \equiv (p_1 + p_2)^2$

Constrains **7** of **11** ISPs

$$(q_1 \cdot q_1) = 0\,, \quad (q_1 \cdot q_2) = 0\,, \quad (q_1 \cdot p_1) = 0\,, \quad (q_1 \cdot p_2) = -\frac{s}{2}\,, \quad (q_2 \cdot q_2) = 0\,,$$
$$(q_2 \cdot p_2) = \frac{s}{2} - (q_2 \cdot p_1)\,, \quad (q_2 \cdot p_3) = -\frac{s}{2}$$

- The more D_i's are set to zero, the more ISP's are constrained

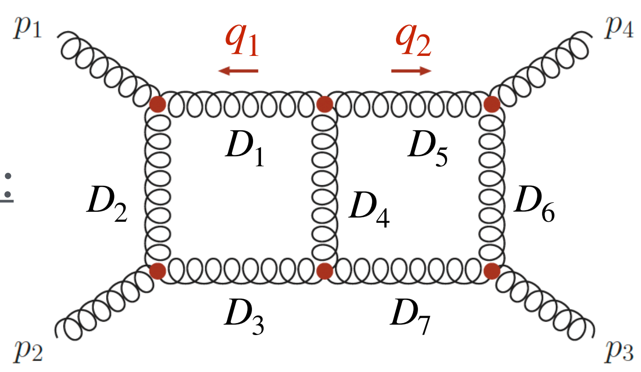
↪ Polynomial $P^{(m)}$ is simpler than $P_i^{(m-1)}$, which in turn is simpler than $P_{ij}^{(m-2)}$ etc …

Projecting over the full family of propagators

$$N(\bar{q}_1, \bar{q}_2, \{p\}) = P^{(m)} + \sum_{i=1}^{9} P_i^{(m-1)} D_i + \sum_{i<j}^{9} P_{ij}^{(m-2)} D_i D_j + \sum_{i<j<k}^{9} P_{ij}^{(m-3)} D_i D_j D_k + \cdots$$

- **Definition of *family*:** enlarged set of D_i's such that *all scalar products* (including ISP's) can be expressed as combinations of them.

e.g.:



"Family" of propagators (9 in total)

$$D_1 = q_1^2, \quad D_2 = (q_1 + p_1)^2, \quad D_3 = (q_1 + p_{12})^2, \quad D_4 = (q_1 + q_2)^2, \quad D_5 = q_2^2,$$
$$D_6 = (q_2 - p_{123})^2, \quad D_7 = (q_2 - p_{12})^2, \quad D_8 = (q_2 - p_1)^2, \quad D_9 = (q_1 - p_{123})^2$$

- Maximal cut equation: $D_1 = D_2 = D_3 = D_4 = D_5 = D_6 = D_7 = D_8 = D_9 = 0$ $\qquad s \equiv (p_1 + p_2)^2$

Constrains **9** of **11** ISPs

$$(q_1 \cdot q_1) = 0, \quad (q_1 \cdot q_2) = 0, \quad (q_1 \cdot p_1) = 0, \quad (q_1 \cdot p_2) = -\frac{s}{2}, \quad (q_2 \cdot q_2) = 0,$$
$$(q_2 \cdot p_2) = \frac{s}{2}, \quad (q_2 \cdot p_1) = 0, \quad (q_2 \cdot p_3) = -\frac{s}{2}, \quad (q_1 \cdot p_3) = \frac{s}{2}$$

- Projecting over *family* helps reducing the complexity of *individual* polynomials (at the cost of more cut equations)

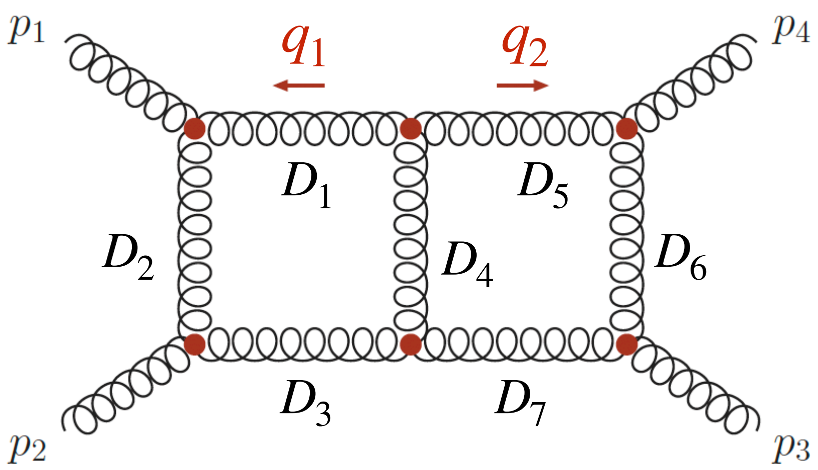
Note: different diagram topologies share the same family

Comparing different cutting approaches

What's the impact on the size of polynomials? Check out *double-box* topology:

"standard cut" approach

Level	Number of cuts	Number of coefficients
7	1	70
6	7	695
5	21	1430
4	35	1017
3	35	225
2	21	9



projecting over family

Level	Number of cuts	Number of coefficients
9	1	13
8	9	227
7	36	963
6	84	1445
5	126	780
4	126	116

- Tradeoff between number of cut equations and coefficients for each polynomial

[G.B, Canko, Spourdalakis and Papadopoulos, 2506.07231 [hep-ph]]

Approaching cut solutions numerically

- The sketched method is independent of dimensionality (works as well in $d = 4$ and in $d = 4 - 2\varepsilon$ dimensions)

- If the Numerator is computed *numerically*, we need to build explicit solutions for $\bar{q}_1, \bar{q}_2$

$N = N(\bar{q}_1, \bar{q}_2, \{p\})$

Dimensional regularisation ⟶ 't Hooft-Veltman scheme

- $p_1, \ldots, p_n$ (ext. momenta) in $d = 4$
- $\bar{q}_1, \bar{q}_2$ (loop momenta) in $d = 4 - 2\varepsilon$

↪ $\bar{q}_1, \bar{q}_2$ have $8 + 3 = \boxed{11}$ d.o.f.

$$\begin{cases} \bar{q}_1^2 = q_1^2 + \mu_{11} \\ \bar{q}_2^2 = q_2^2 + \mu_{22} \\ (\bar{q}_1 \cdot \bar{q}_2) = (q_1 \cdot q_2) + \mu_{12} \end{cases}$$

- Working in $d = 4 - 2\varepsilon$, there is sufficient freedom to ensure that cut equations, at any level, admit a *unique* parametric solution for $\bar{q}_1, \bar{q}_2$:

$$D_1 = D_2 = \cdots D_i = 0 \quad \Rightarrow \quad q_1 = q_1(\{p\}, \vec{x}), \quad q_2 = q_2(\{p\}, \vec{y}), \quad \mu_{11}, \mu_{12}, \mu_{22}$$

[$\vec{x}, \vec{y}, \mu_{ij}$ = free parameters]

What if we work in $d = 4$? $^{(*)}$

$^{(*)}$ R_2 rational terms must to be provided at some later stage
[Pozzorini, Lang, Zhang, Zoller '20, '22]

- In $d = 4$, cut solutions do *not* admit a unique parametric solution in general, and are structured in different *branches*:

$$D_1 = D_2 = \cdots D_i = 0 \quad \rightarrow \quad \left[\begin{array}{ll} \left\{ q_1^{(1)}(\{p\}, \vec{x}), q_2^{(1)}(\{p\}, \vec{y}) \right\} & \longrightarrow \quad 1^{\text{st}} \text{ branch} \\ \vdots & \qquad\qquad \vdots \\ \left\{ q_1^{(r)}(\{p\}, \vec{x}), q_2^{(r)}(\{p\}, \vec{y}) \right\} & \longrightarrow \quad r^{\text{th}} \text{ branch} \end{array} \right.$$

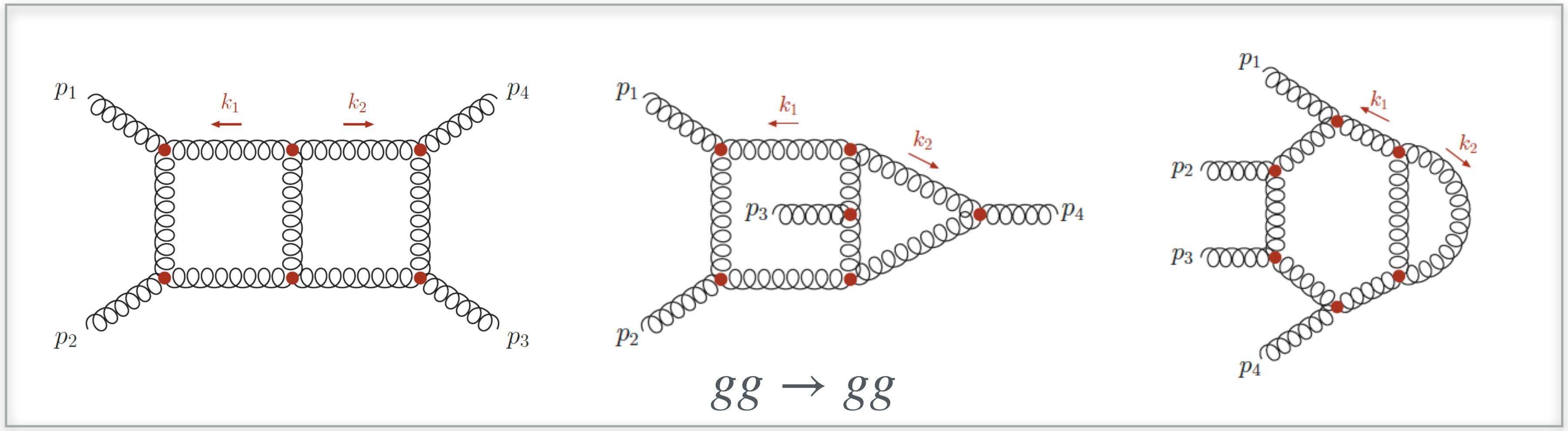
- Solutions from *all* branches need to be considered for fitting coefficients!
  ↪ larger system of equations to solve (more rows)

$$\begin{pmatrix} N_1^{(1)} \\ \vdots \\ N_m^{(r)} \end{pmatrix} = \begin{pmatrix} M_{1,1} & \cdots & M_{1,m} \\ & \ddots & \\ M_{m\cdot r,1} & \cdots & M_{m\cdot r,m} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \quad \leftrightarrow \quad \vec{N} = \mathbf{M} \cdot \vec{c}$$

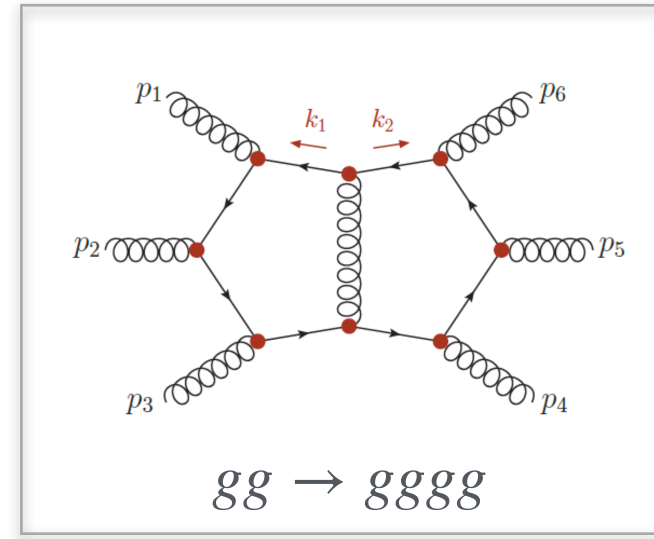
- There are cases where the matrix $\mathbf{M}$ is rank-deficient

Validation of the method

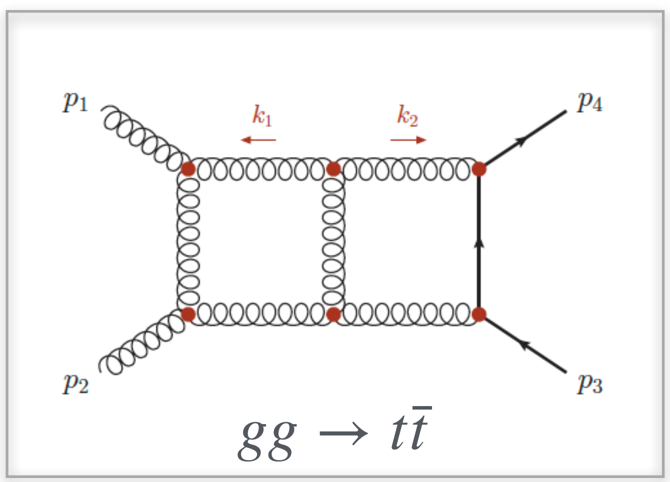
- We have validated our method considering up to 6-point loop functions, with *massless* and *massive* propagators



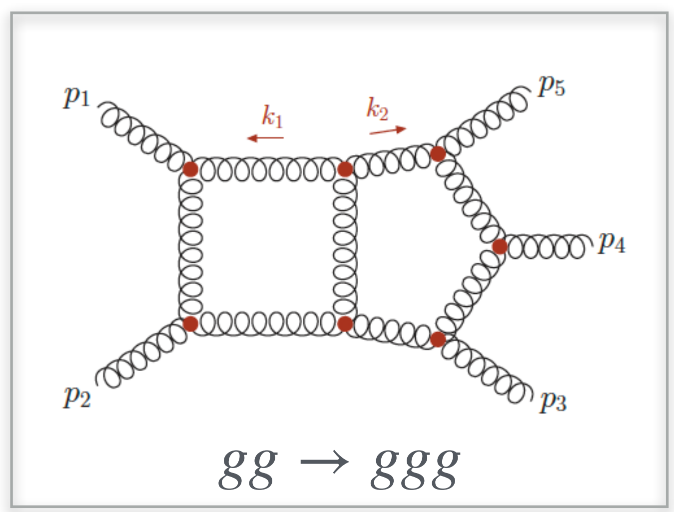
$gg \to gg$



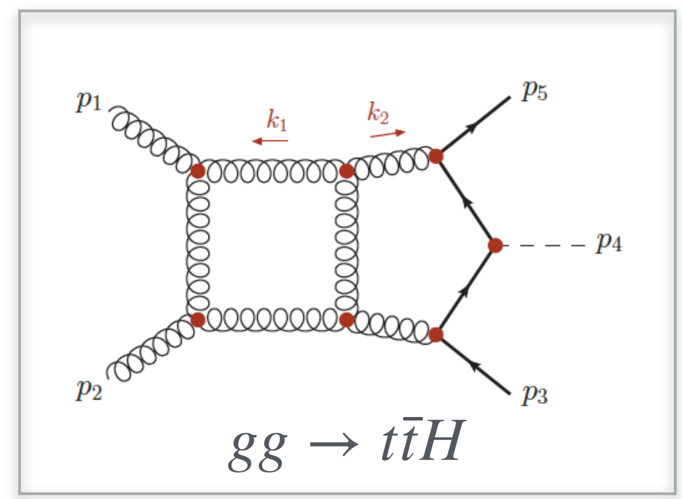
$gg \to gggg$



$gg \to t\bar{t}$



$gg \to ggg$



$gg \to t\bar{t}H$

- $N = N$ checks performed in both $d = 4$ and $d = 4 - 2\varepsilon$ dimensions

[G.B, Canko, Spourdalakis and Papadopoulos, 2506.07231 [hep-ph]]

Summary and conclusions

- We have established a *method* to fit coefficients of the reduction at integrand level
  - the method works in either $d = 4$ or $d = 4 - 2\varepsilon$ dimensions
  - addressing IBP reduction of Feynman integrals is the next big step

- Performing the reduction in $d = 4 - 2\varepsilon$ dimensions potentially pays off
  - no need to separately compute R_2 rational terms
  - simpler structure of cut solutions $(\bar{q}_1, \bar{q}_2)$

- How feasible are numerical computations of loop numerators in $d = 4 - 2\varepsilon$?

  Requisites:
  - a procedure to supplement terms $\propto \mu_{ij}, \varepsilon$ to numerical computations of numerators;
  - a "numerator provider" working in $d = 4 - 2\varepsilon$ dimensions (possibly general-purpose);
  - a ready-to-use implementation of the reduction algorithm (such as e.g. CutTools).

  ↪ these steps are essential to assess the numerical performance of the method

Outlook (I)

Towards numerical computations of dimensionally regularised Numerators

G.B, Canko, Papadopoulos and Spourdalakis, in progress
[see also Fazio, Mastrolia, Mirabella and Torres Bobadilla '14]

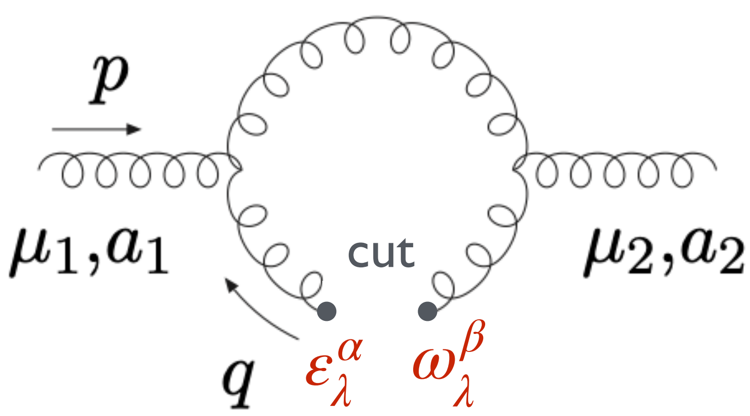
Simple example: QCD @ 1-loop (only gluons)

- μ and ε terms arise upon contracting few Lorentz structures: $\tilde{q}^{\alpha}$, $\tilde{\varepsilon}_{\lambda}{}^{\alpha}$, $\tilde{\omega}_{\lambda}{}^{\alpha}$, $(\tilde{q}\cdot\tilde{\varepsilon}_{\lambda})\,\tilde{q}^{\alpha}$

$$q^2 X \to \mu X$$

$$\sum_{\lambda} (\varepsilon_{\lambda}\cdot\omega_{\lambda})\, X \equiv g^{\mu\nu} g_{\mu\nu}\, X \to (d-4)\, X$$

$$\sum_{\lambda} (\varepsilon_{\lambda}\cdot q)(\omega_{\lambda}\cdot q)\, X \equiv q^2 X \to \mu X$$



$$\hookrightarrow \sum_{\lambda} \varepsilon_{\lambda}^{\alpha}\omega_{\lambda}^{\beta} \equiv g^{\alpha\beta}$$

- Track the *coefficients* of such structures appearing in the current J at each step of recursion:

$$J^{(N)\,\alpha} \to J^{(n)\,\alpha} + \boxed{C_q^{(N)}}\,\tilde{q}^{\alpha} + \boxed{C_{\varepsilon}^{(N)}}\,\tilde{\varepsilon}_{\lambda}{}^{\alpha} + \boxed{J_{q\varepsilon}^{(N)\,\alpha}}(\tilde{q}\cdot\tilde{\varepsilon}_{\lambda}) + \boxed{C_{q\varepsilon q}^{(N)}}(\tilde{q}\cdot\tilde{\varepsilon}_{\lambda})\,\tilde{q}^{\alpha}$$

- Derive recursion relations for the coefficients;
- At each step of the recursion, compute and add μ and ε terms to the current.

Towards numerical computations of dimensionally regularised Numerators

G.B, Canko, Papadopoulos and Spourdalakis, in progress

[see also Fazio, Mastrolia, Mirabella and Torres Bobadilla '14]

Simple exam

- μ and ε t ... $_\lambda)\,\tilde{q}^\alpha$

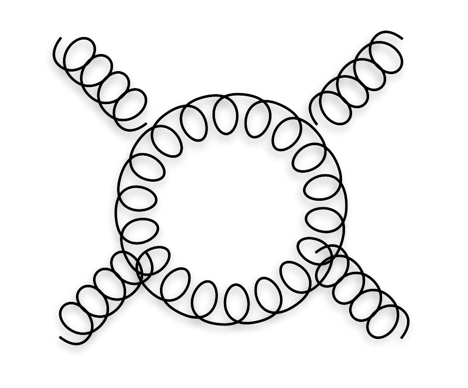
a_2

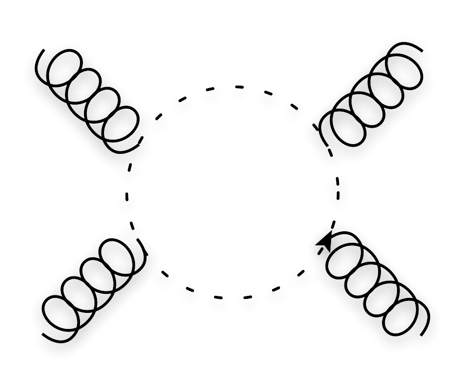
$\omega_\lambda^\beta \equiv g^{\alpha\beta}$

- Track th ... recursion:

$J^{(N}$ ... $_\lambda)\,\tilde{q}^\alpha$

I. Completed 1-loop numerical validation $(gg \to gg)$





- Cut Constructible (CC) + Rational Terms computed at *integrand* level

vs

- *Standard approach*: CC+R_1 at integrand level, R_2 via effective Feynman rules

- Derive recursion relations for the coefficients;
- At each step of the recursion, compute and add μ and ε terms to the current.

Towards numerical computations of dimensionally regularised Numerators

G.B, Canko, Papadopoulos and Spourdalakis, in progress
[see also Fazio, Mastrolia, Mirabella and Torres Bobadilla '14]

Simple examp

- μ and ε t ... $_\lambda)\,\tilde{q}^\alpha$

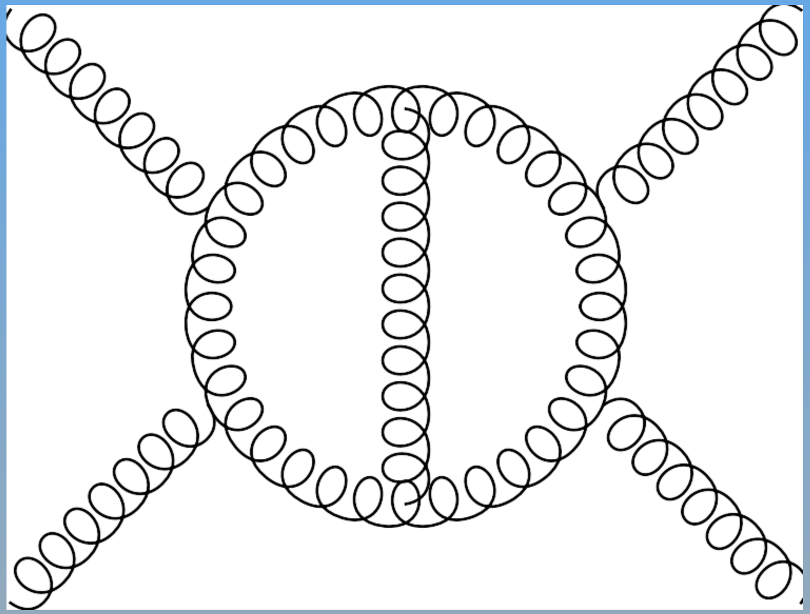
a_2

$\omega_\lambda^\beta \equiv g^{\alpha\beta}$

- Track th ... recursion:

$J^{(N}$ ... $_\lambda)\,\tilde{q}^\alpha$

II. Completed 2-loop analytic proof of concept
($gg \to gg$ - double-box topology)



- all μ_{ij} and ε terms reproduced successfully

- Derive recursion relations for the coefficients;
- At each step of the recursion, compute and add μ and ε terms to the current.

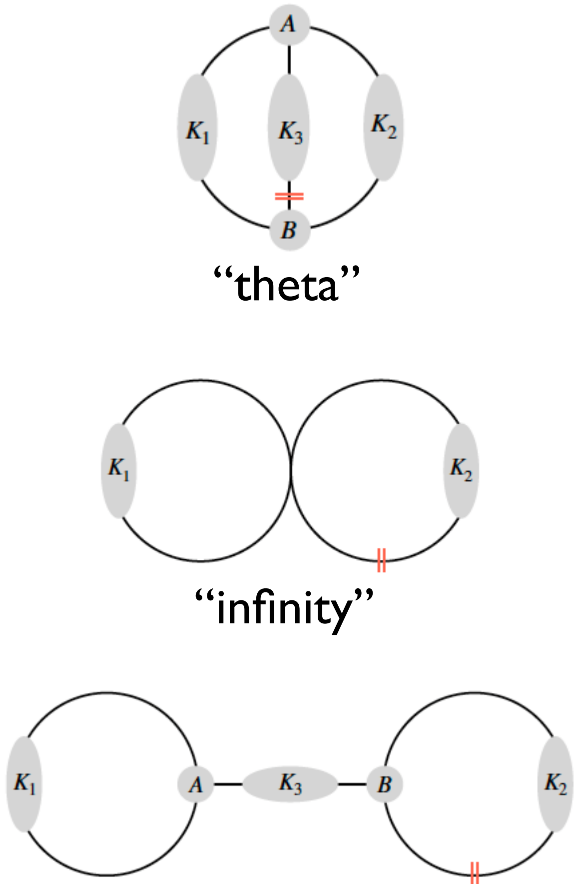
Constructing HELAC-2LOOP

G.B, Canko, Papadopoulos and Spourdalakis, in progress

- General algorithm

{G.B, Canko and Papadopoulos, PoS RADCOR2023 (2024) 081]

Master 2-loop topologies

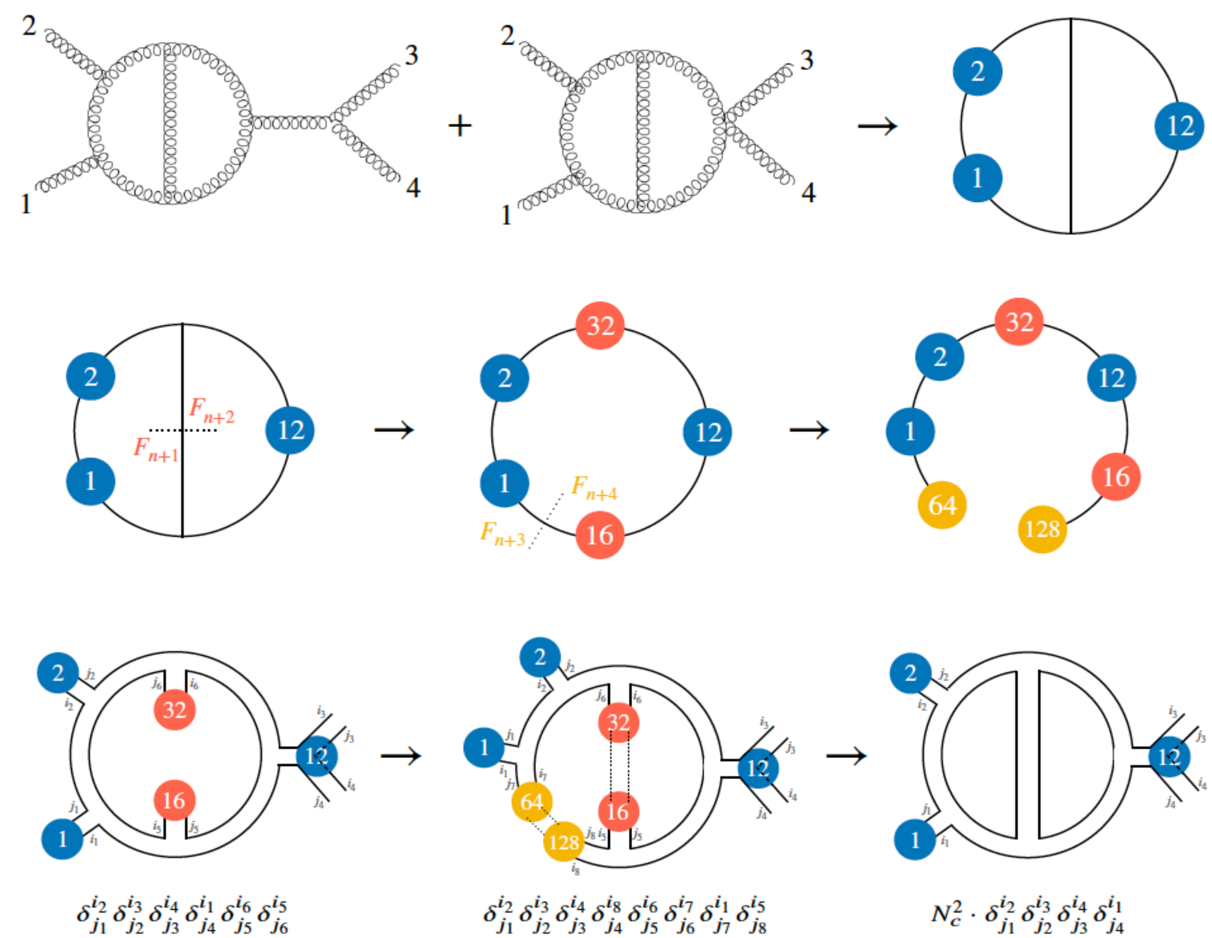


"theta"

"infinity"

"dumbbell"

2-loop numerator computation strategy



- Proof of concept ($d = 4 - 2\varepsilon$) & code implementing the reduction under construction

Thank you for your attention

Ευχαριστώ για τη προσοχή σας

Backup slides

2-loop reduction: a toy example

$$A_{2L} = \int d^d\bar{q}_1\, d^d\bar{q}_2 \frac{N(\bar{q}_1, \bar{q}_2, \{p\})}{\prod_{i=1}^{7} D_i(\bar{q}_1, \bar{q}_2, \{p\})}$$

$$N(\bar{q}_1, \bar{q}_2, \{p\}) \equiv (\bar{q}_1 \cdot \bar{q}_2)(p_2 \cdot p_3) + (\bar{q}_2 \cdot p_4)(p_1 \cdot p_2)$$

2-loop reduction: a toy example

$$A_{2L} = \int d^d\bar{q}_1\, d^d\bar{q}_2 \frac{N(\bar{q}_1, \bar{q}_2, \{p\})}{\prod_{i=1}^{7} D_i(\bar{q}_1, \bar{q}_2, \{p\})}$$

$$N(\bar{q}_1, \bar{q}_2, \{p\}) \equiv (\bar{q}_1 \cdot \bar{q}_2)(p_2 \cdot p_3) + (\bar{q}_2 \cdot p_4)(p_1 \cdot p_2)$$

$$= -\frac{1}{2}(D_7 - D_1 - D_2)(p_2 \cdot p_3) + (\bar{q}_2 \cdot p_4)(p_1 \cdot p_2)$$

$$\begin{aligned}
D_1 &= \bar{q}_1{}^2 \\
D_2 &= \bar{q}_2{}^2 \\
D_3 &= (\bar{q}_2 + p_2)^2 \\
D_4 &= (\bar{q}_2 + p_2 + p_3)^2 \\
D_5 &= (\bar{q}_1 + p_2 + p_3)^2 \\
D_6 &= (\bar{q}_1 + p_2 + p_3 + p_4)^2 \\
D_7 &= (\bar{q}_1 - \bar{q}_2)^2
\end{aligned}$$

- Part of the above scalars can be *reduced* in terms of the D_i's appearing in the Denominator: *Reducible Scalar Products* (RSP)

2-loop reduction: a toy example

$$A_{2L} = \int d^d\bar{q}_1\, d^d\bar{q}_2 \, \frac{N(\bar{q}_1, \bar{q}_2, \{p\})}{\prod_{i=1}^{7} D_i(\bar{q}_1, \bar{q}_2, \{p\})}$$

$$\begin{aligned} N(\bar{q}_1, \bar{q}_2, \{p\}) &\equiv (\bar{q}_1 \cdot \bar{q}_2)(p_2 \cdot p_3) + (\bar{q}_2 \cdot p_4)(p_1 \cdot p_2) \\ &= -\frac{1}{2}(D_7 - D_1 - D_2)(p_2 \cdot p_3) + (\bar{q}_2 \cdot p_4)(p_1 \cdot p_2) \end{aligned}$$

$$\begin{aligned} D_1 &= \bar{q}_1{}^2 \\ D_2 &= \bar{q}_2{}^2 \\ D_3 &= (\bar{q}_2 + p_2)^2 \\ D_4 &= (\bar{q}_2 + p_2 + p_3)^2 \\ D_5 &= (\bar{q}_1 + p_2 + p_3)^2 \\ D_6 &= (\bar{q}_1 + p_2 + p_3 + p_4)^2 \\ D_7 &= (\bar{q}_1 - \bar{q}_2)^2 \end{aligned}$$

- Part of the above scalars can be *reduced* in terms of the D_i's appearing in the Denominator: *Reducible Scalar Products* (RSP)
- The subset of scalars that cannot be decomposed this way are *Irreducible Scalar Products* (ISP)

2-loop reduction: a toy example

$$A_{2L} = \int d^d\bar{q}_1\, d^d\bar{q}_2\, \frac{N(\bar{q}_1, \bar{q}_2, \{p\})}{\prod_{i=1}^{7} D_i(\bar{q}_1, \bar{q}_2, \{p\})}$$

$$N(\bar{q}_1, \bar{q}_2, \{p\}) \equiv (\bar{q}_1 \cdot \bar{q}_2)(p_2 \cdot p_3) + (\bar{q}_2 \cdot p_4)(p_1 \cdot p_2)$$

$$= -\frac{1}{2}(D_7 - D_1 - D_2)(p_2 \cdot p_3) + (\bar{q}_2 \cdot p_4)(p_1 \cdot p_2)$$

$$D_1 = \bar{q}_1^{\,2}$$
$$D_2 = \bar{q}_2^{\,2}$$
$$D_3 = (\bar{q}_2 + p_2)^2$$
$$D_4 = (\bar{q}_2 + p_2 + p_3)^2$$
$$D_5 = (\bar{q}_1 + p_2 + p_3)^2$$
$$D_6 = (\bar{q}_1 + p_2 + p_3 + p_4)^2$$
$$D_7 = (\bar{q}_1 - \bar{q}_2)^2$$

$$A_{2L} = \int d\bar{q}_1\, d\bar{q}_2 \left\{ \frac{-(p_2 \cdot p_3)/2}{D_1 D_2 D_3 D_4 D_5 D_6} + \frac{(p_2 \cdot p_3)/2}{D_2 D_3 D_4 D_5 D_6 D_7} + \frac{(p_2 \cdot p_3)/2}{D_1 D_3 D_4 D_5 D_6 D_7} + \frac{(\bar{q}_2 \cdot p_4)(p_1 \cdot p_2)}{D_1 D_2 D_3 D_4 D_5 D_6 D_7} \right\}$$

- Part of the above scalars can be *reduced* in terms of the D_i's appearing in the Denominator: *Reducible Scalar Products* (RSP)
- The subset of scalars that cannot be decomposed this way are *Irreducible Scalar Products* (ISP)
- After reduction, the 2-loop amplitude is a combination of *simpler* Feynman integrals