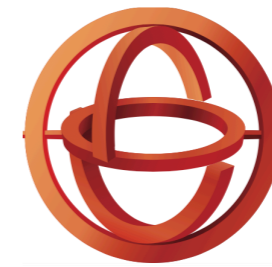




NATIONAL CENTRE FOR  
SCIENTIFIC RESEARCH "DEMOKRITOS"  
INSTITUTE OF NUCLEAR AND PARTICLE PHYSICS



**H.F.R.I.**  
Hellenic Foundation for  
Research & Innovation

# **Numerical evaluation of dimensionally regularised amplitudes**

**Giuseppe Bevilacqua**  
NCSR "Demokritos"

2nd HOCTools-II mini-workshop on Higher Order  
Calculations and Tools for High Energy Colliders

NCSR "Demokritos", October 27, 2025

In collaboration with C. Papadopoulos, D. Canko and A. Spourdalakis

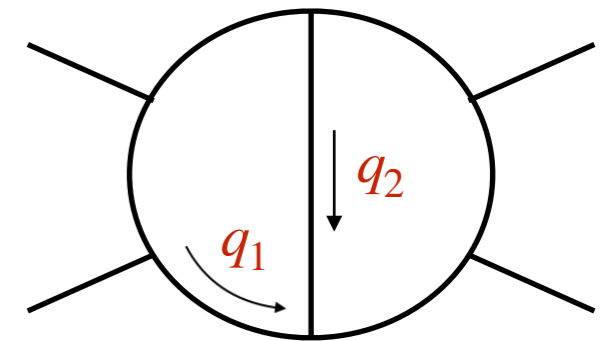
*Work in progress*

# Introduction

- We have presented a method for extracting coefficients of two-loop reduction at **integrand level**
  - The method works in either  $d = 4$  or  $d = 4 - 2\varepsilon$  dimensions
  - Addressing IBP reduction of Feynman integrals is the next big step

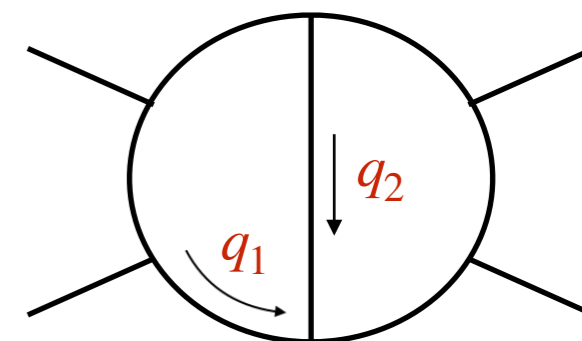
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  - No need to compute  $R_2$  rational terms separately



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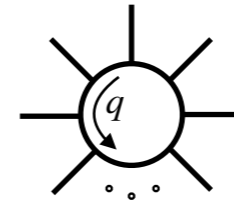
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- Performing reduction in  $d = 4 - 2\varepsilon$  potentially pays off
  - Solutions of cut equations get simpler structure
  - No need to compute  $R_2$  rational terms separately
- How feasible are **recursive** computations of loop numerators in  $d = 4 - 2\varepsilon$  ?
  - ↪ Requisites:
    - a procedure to supplement **evanescent terms** at any step of the recursion;
    - a “numerator provider” working in  $d = 4 - 2\varepsilon$  dimensions (possibly general-purpose)



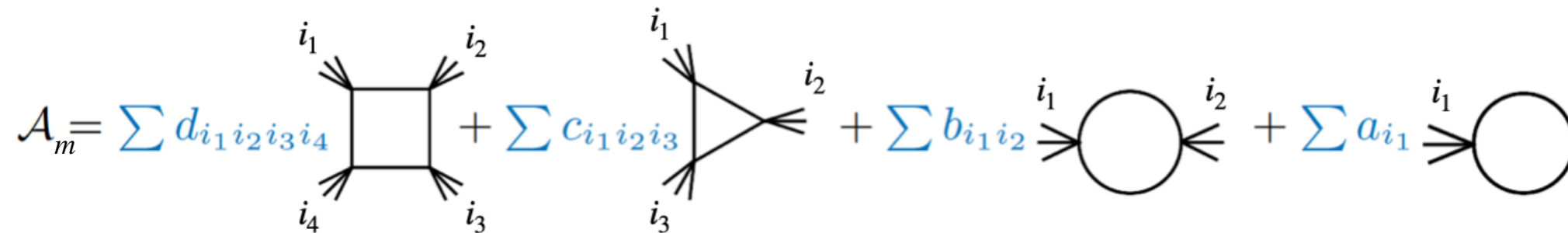
# Recap: origin of Rational Terms

## I-loop OPP reduction in a nutshell

{Ossola, Papadopoulos and Pittau, [Nucl.Phys.B 763 \(2007\) 147-169](#)}



$$\mathcal{A}_m = \int \frac{d^d q}{(2\pi)^d} \frac{\bar{N}(q)}{\bar{D}_0 \bar{D}_1 \dots \bar{D}_{m-1}}$$

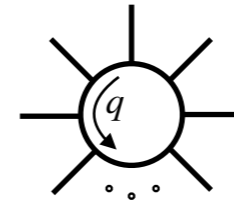
$$\mathcal{A}_m = \sum d_{i_1 i_2 i_3 i_4} \text{ (square diagram)} + \sum c_{i_1 i_2 i_3} \text{ (triangle diagram)} + \sum b_{i_1 i_2} \text{ (circle diagram)} + \sum a_{i_1} \text{ (circle diagram)}$$


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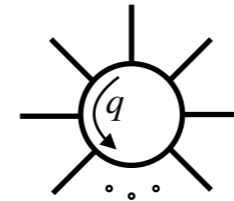
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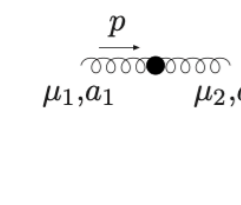


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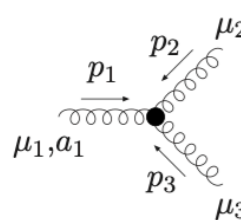
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- Typically, numerical ME generators compute numerators in  $d = 4$
- However in *dimensional regularisation*, the loop integrand is defined in  $d = 4 - 2\epsilon$
- The mismatch can be compensated by adding Rational Terms ( $R_1, R_2$ ) to the computation
- $R_2$  can be cast in the form of extra **tree-level Feynman rules**

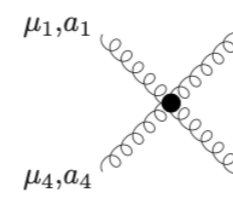
example: QCD  
[Draggiotis et al, '09]



$$= \frac{ig^2 N_{\text{col}}}{48\pi^2} \delta_{a_1 a_2} \left[ \frac{p^2}{2} g_{\mu_1 \mu_2} + \lambda_{\text{HV}} (g_{\mu_1 \mu_2} p^2 - p_{\mu_1} p_{\mu_2}) + \frac{N_f}{N_{\text{col}}} (p^2 - 6 m_q^2) g_{\mu_1 \mu_2} \right]$$



$$= -\frac{g^3 N_{\text{col}}}{48\pi^2} \left( \frac{7}{4} + \lambda_{\text{HV}} + 2 \frac{N_f}{N_{\text{col}}} \right) f^{a_1 a_2 a_3} V_{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$



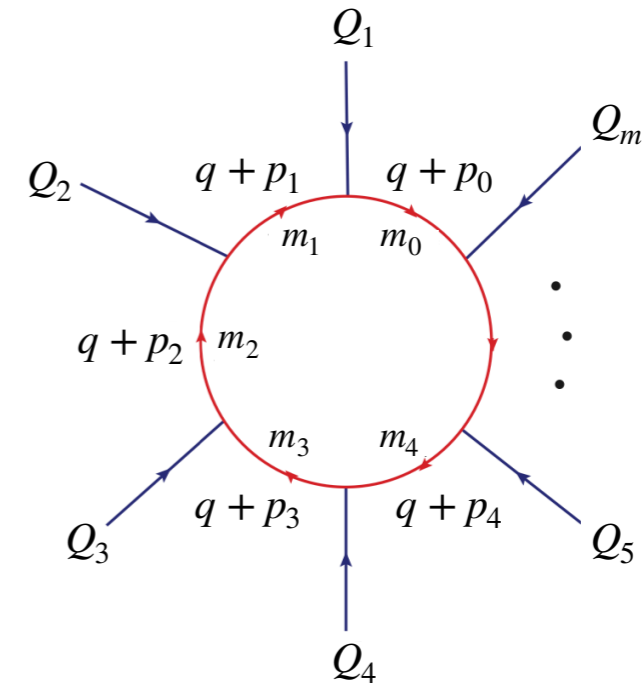
$$= -\frac{ig^4 N_{\text{col}}}{96\pi^2} \sum_{P(234)} \left\{ \left[ \frac{\delta_{a_1 a_2} \delta_{a_3 a_4} + \delta_{a_1 a_3} \delta_{a_4 a_2} + \delta_{a_1 a_4} \delta_{a_2 a_3}}{N_{\text{col}}} + 4 \text{Tr}(t^{a_1} t^{a_3} t^{a_2} t^{a_4} + t^{a_1} t^{a_4} t^{a_2} t^{a_3}) (3 + \lambda_{\text{HV}}) - \text{Tr}(\{t^{a_1} t^{a_2}\} \{t^{a_3} t^{a_4}\}) (5 + 2\lambda_{\text{HV}}) \right] g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} + 12 \frac{N_f}{N_{\text{col}}} \text{Tr}(t^{a_1} t^{a_2} t^{a_3} t^{a_4}) \left( \frac{5}{3} g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} - g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} - g_{\mu_2 \mu_3} g_{\mu_1 \mu_4} \right) \right\}$$

# Basics of t'Hooft-Veltman regularisation

## Dimensional regularisation (**t Hooft-Veltman scheme**)

$$\mathcal{A}_m = \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \dots \bar{D}_{m-1}}$$

- physical momenta ( $Q_1, Q_2, \dots, Q_m$ ) in  $d = 4$
- loop momentum ( $\bar{q}$ ) in  $d = 4 - 2\varepsilon$

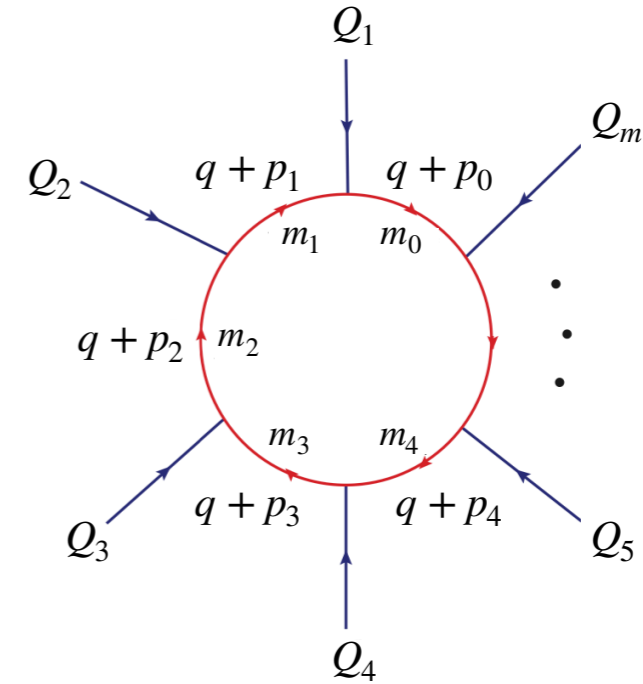


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$$d = 4 - 2\epsilon$$

$$\bar{q}^2 = q^2 + \underbrace{\tilde{q}^2}_{\mu}$$

$$\bar{\gamma}^\mu = \gamma^\mu + \tilde{\gamma}^\mu \quad \hookrightarrow \quad \bar{\gamma}^\mu \bar{\gamma}_\mu = d = 4 - 2\epsilon$$

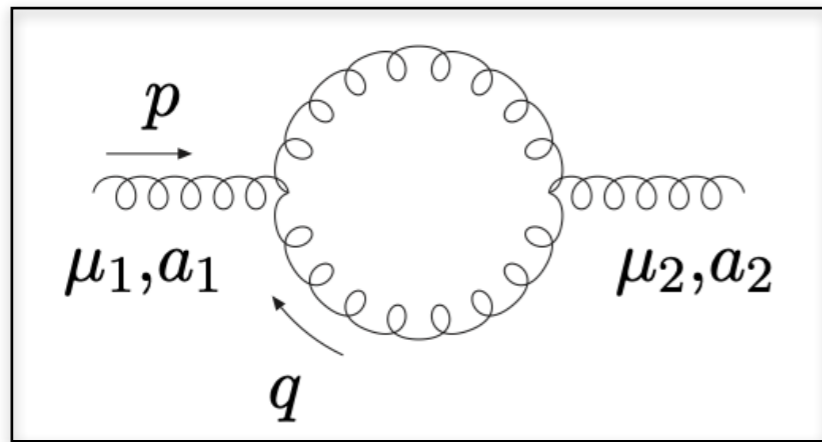
$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu} \quad \hookrightarrow \quad \bar{g}^{\mu\nu} \bar{g}_{\mu\nu} = d = 4 - 2\epsilon$$

$$\{\gamma^\alpha, \gamma^\beta\} = 2 g^{\alpha\beta} I_4$$

$$\{\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta\} = 2 \tilde{g}^{\alpha\beta} I_4$$

$$\{\gamma^\alpha, \tilde{\gamma}^\beta\} = 0$$

# Origin of evanescent terms: a simple example



$$= g f^{a_1 a_2 a_3} \underbrace{V_{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)}_{\text{III}}$$

$$= g f^{a_1 a_2 a_3} \left[ g_{\mu_1 \mu_2} (p_2 - p_1)_{\mu_3} + g_{\mu_2 \mu_3} (p_3 - p_2)_{\mu_1} + g_{\mu_3 \mu_1} (p_1 - p_3)_{\mu_2} \right]$$

- Loop numerator in  $d = 4 - 2\epsilon$  has a genuine **4-dimensional** and an **evanescent** component:

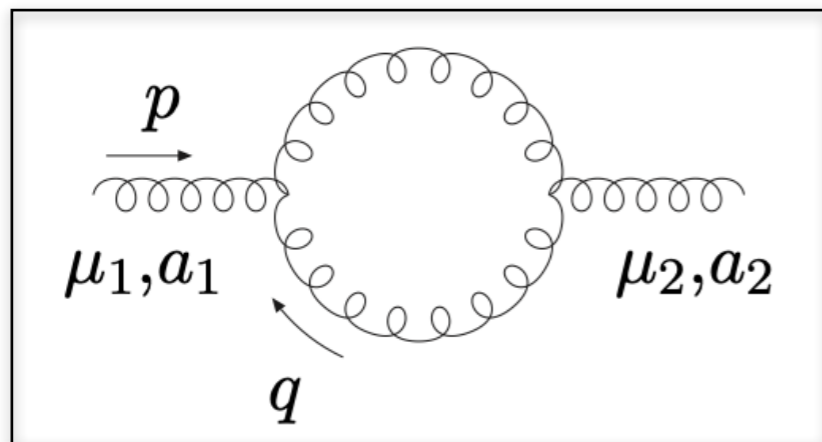
$$\bar{N}(\bar{q}) = \bar{N}^{\mu_1 \mu_2}(\bar{q}) \varepsilon_{\mu_1}(p) \varepsilon_{\mu_2}(p)$$

$$\hookrightarrow \bar{N}^{\mu_1 \mu_2}(\bar{q}) \rightarrow V_{\bar{\beta}\bar{\gamma}}^{\mu_1}(p, -\bar{q}-p, \bar{q}) V^{\mu_2 \bar{\gamma}\bar{\beta}}(-p, -\bar{q}, \bar{q}+p) =$$

$$= (5p^2 + 2p \cdot q + 2q^2) g^{\mu_1 \mu_2} - 2p^{\mu_1} p^{\mu_2} + 5p^{\mu_1} q^{\mu_2} + 5q^{\mu_1} p^{\mu_2} + 10q^{\mu_1} q^{\mu_2}$$

$$- \epsilon (2p^{\mu_1} p^{\mu_2} + 4p^{\mu_1} q^{\mu_2} + 4q^{\mu_1} p^{\mu_2} + 8q^{\mu_1} q^{\mu_2}) + 2\mu g^{\mu_1 \mu_2}$$

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$$\bar{N}(\bar{q}) = \bar{N}^{\mu_1 \mu_2}(\bar{q}) \varepsilon_{\mu_1}(p) \varepsilon_{\mu_2}(p) \equiv N(q) + \tilde{N}(q)$$

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The evanescent component of the numerator,  $\tilde{N}(q)$ , is made of terms  $\propto \mu, \epsilon$

# Computing evanescent terms

$$\mathcal{A}_m = \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \dots \bar{D}_{m-1}} \rightarrow N(q) + \tilde{N}(q)$$

- Can we compute  $\tilde{N}(q)$  using a *numerical, recursive* framework (built in  $d = 4$ )?

schematically:

$$\tilde{N}(q) = \mathcal{E}[N(q)]$$

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- Possible solution: *Four-Dimensional Formulation (FDF)*

[ Fazio, Mastrolia, Mirabella and Torres Bobadilla, [Eur.Phys.J.C 74 \(2014\) 12, 3197](#) ]

- ↳ • developed for 1-loop calculations
- 4-dimensional d.o.f of gauge bosons carried out by  $\mu$
- $(d - 4)$  dimensional d.o.f carried out by *scalar* particles  $\Rightarrow$  *extra Feynman rules*

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- Can we compute  $\tilde{N}(q)$  (and  $\tilde{D}_i$ ) in  $d = 4$ ?

Fazio, Mastrolia, Mirabella and Torres Bobadilla, *Eur.Phys.J.C* 74 (2014) 12, 3197

- Possible sources

- developed in
- 4-dim
- $(d - 2)$

$\text{gluon: } \begin{array}{c} k \\ \text{---} \\ a, \alpha \quad b, \beta \end{array} = -i \delta^{ab} \frac{g^{\alpha\beta}}{k^2 - \mu^2 + i0}$   
 $\text{ghost: } \begin{array}{c} k \\ \text{---} \\ a \quad b \end{array} = i \delta^{ab} \frac{1}{k^2 - \mu^2 + i0}$   
 $\text{scalar: } \begin{array}{c} k \\ \text{---} \\ a, A \quad b, B \end{array} = -i \delta^{ab} \frac{G^{AB}}{k^2 - \mu^2 + i0}$   
 $\text{fermion: } \begin{array}{c} k \\ \text{---} \\ i \quad j \end{array} = i \delta^{ij} \frac{\not{k} + i\mu\gamma^5 + m}{k^2 - m^2 - \mu^2 + i0}$   
 $\text{ghost-gluon: } \begin{array}{c} 2, b, \beta \\ \text{---} \\ 1, a, \alpha \quad 3, c, \gamma \end{array} = -g f^{abc} [(k_1 - k_2)^\gamma g^{\alpha\beta} + (k_2 - k_3)^\alpha g^{\beta\gamma} + (k_3 - k_1)^\beta g^{\gamma\alpha}]$   
 $\text{ghost-gluon: } \begin{array}{c} 2, b \\ \text{---} \\ 1, a, \alpha \quad 3, c \end{array} = -g f^{abc} k_2^\alpha$

$\text{ghost-gluon-gluon: } \begin{array}{c} 2, b, B \\ \text{---} \\ 1, a, \alpha \quad 3, c, C \end{array} = -g f^{abc} (k_2 - k_3)^\alpha G^{BC}$   
 $\text{ghost-gluon-ghost: } \begin{array}{c} 2, b, B \\ \text{---} \\ 1, a, \alpha \quad 3, c, \gamma \end{array} = \mp g f^{abc} (i\mu) g^{\gamma\alpha} Q^B, \quad (\tilde{k}_1 = 0, \tilde{k}_3 = \pm \tilde{\ell})$   
 $\text{ghost-gluon-fermion: } \begin{array}{c} 2, b, \beta \\ \text{---} \\ 1, a, \alpha \quad 3, c, \gamma \end{array} = -ig^2 [ + f^{xad} f^{xbc} (g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) + f^{xac} f^{xbd} (g^{\alpha\beta} g^{\delta\gamma} - g^{\alpha\delta} g^{\beta\gamma}) + f^{xab} f^{xdc} (g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\gamma} g^{\beta\delta}) ]$   
 $\text{ghost-gluon-gluon-gluon: } \begin{array}{c} 2, b, B \\ \text{---} \\ 1, a, \alpha \quad 3, c, C \end{array} = 2ig^2 g^{\alpha\delta} (f^{xab} f^{xcd} + f^{xac} f^{xbd}) G^{BC}$   
 $\text{ghost-gluon-ghost-gluon: } \begin{array}{c} 2, b, \beta \\ \text{---} \\ 1, i \quad 3, j \end{array} = -ig (t^b)_{ji} \gamma^\beta$   
 $\text{ghost-gluon-fermion-gluon: } \begin{array}{c} 2, b, B \\ \text{---} \\ 1, i \quad 3, j \end{array} = -ig (t^b)_{ji} \gamma^5 \Gamma^B$

[14) 12, 3197]

Feynman rules

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[ Fazio, Mastrolia, Mirabella and Torres Bobadilla, [Eur.Phys.J.C 74 \(2014\) 12, 3197](#) ]

- We are exploring an *alternative* formulation (not requiring extra Feynman rules) *and* we want to extend it to 2-loop applications

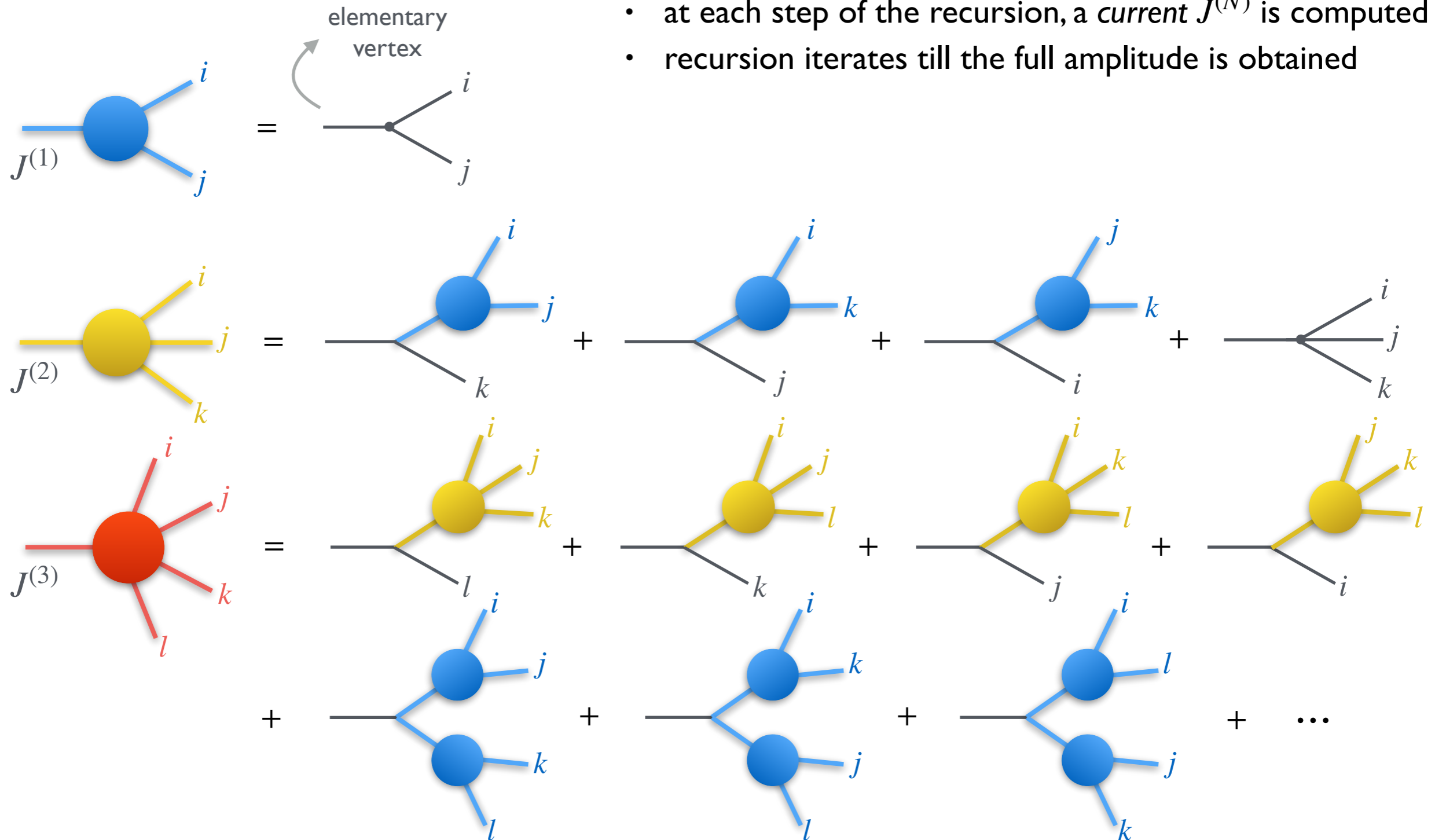
## I. Basic formalism

# **I-loop construction**

# Recursive amplitude calculations

## Sketch of recursive amplitude computation

- at each step of the recursion, a *current*  $J^{(N)}$  is computed
- recursion iterates till the full amplitude is obtained



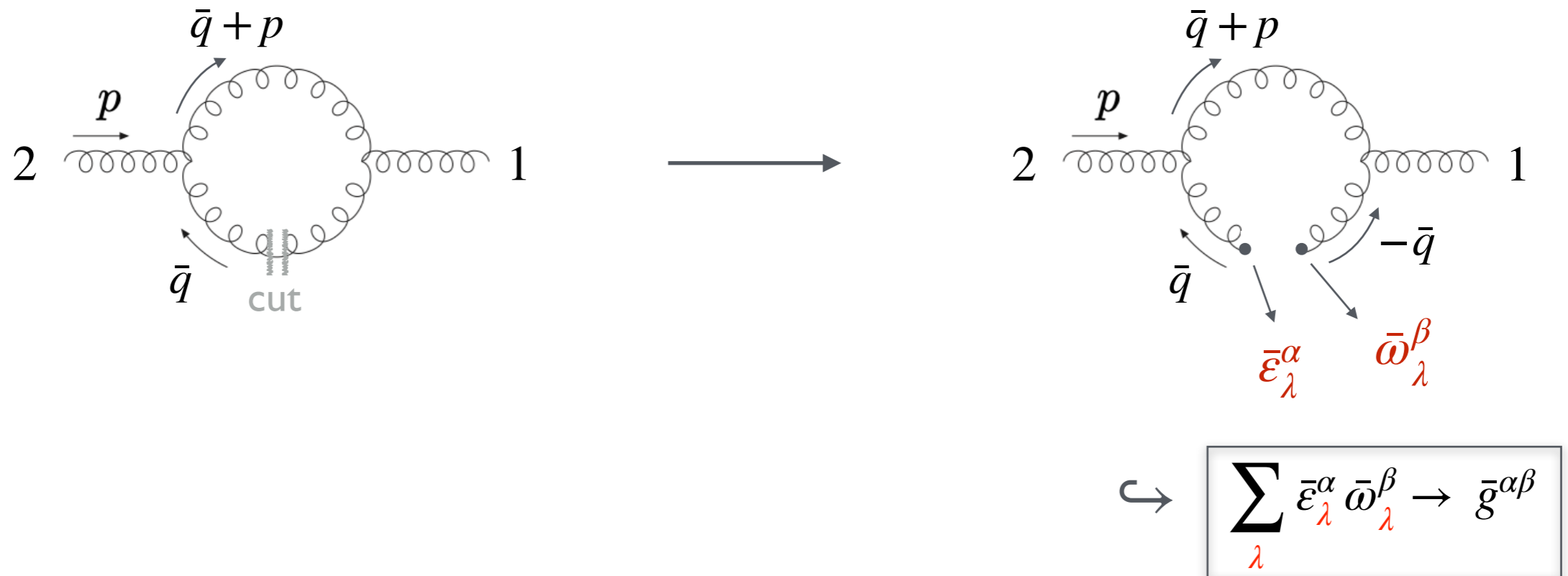
# Computing loop numerators recursively

- **Cutting** loop propagator  $\rightarrow$  Tree-level process with two extra particles



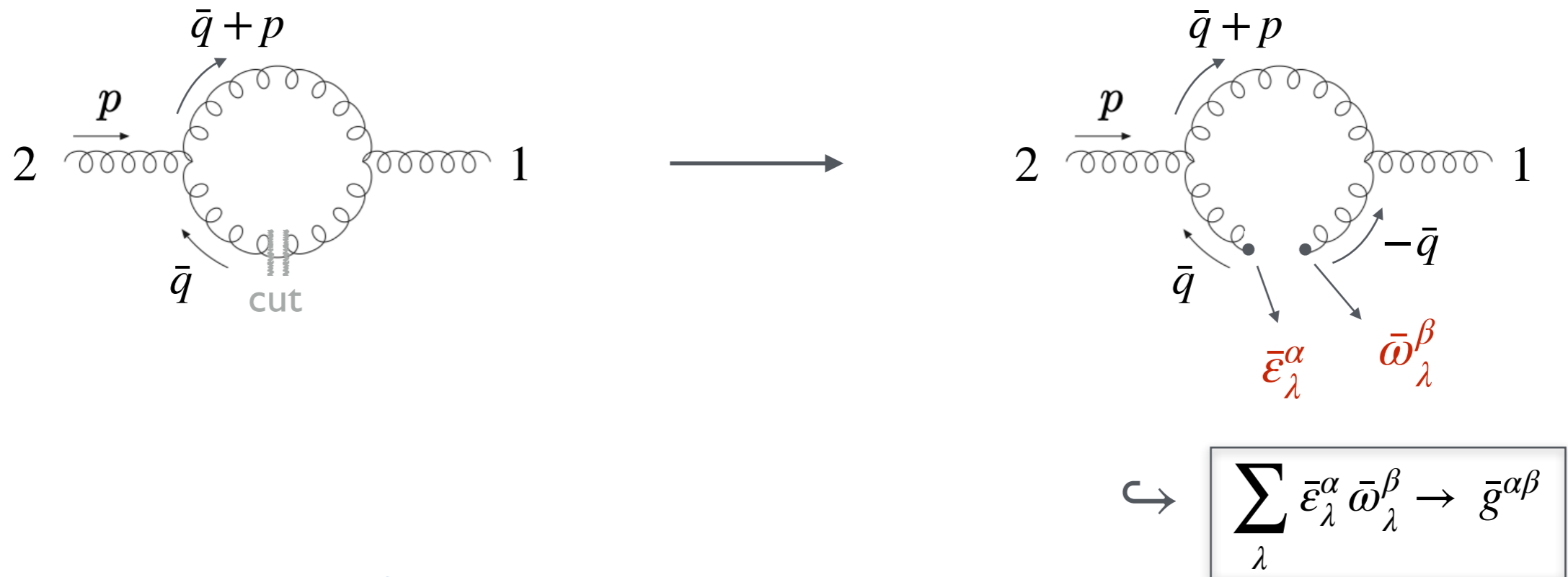
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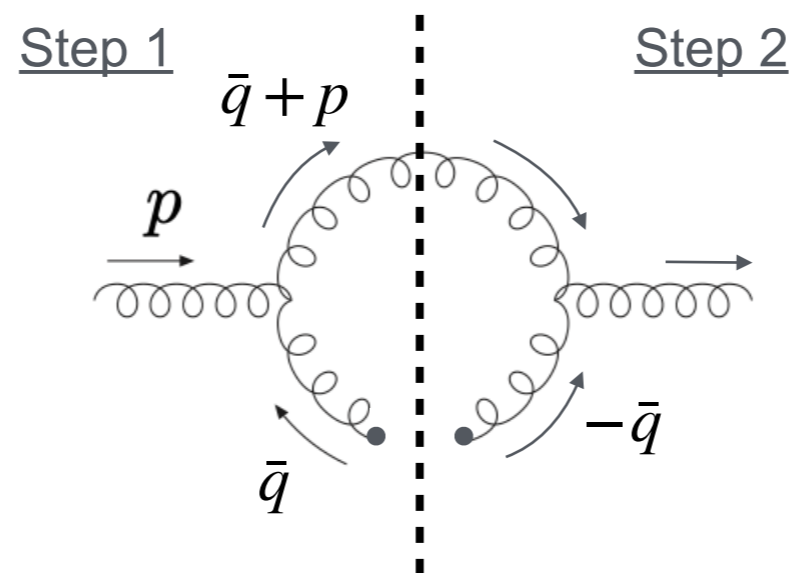


# Computing loop numerators recursively

- **Cutting** loop propagator  $\rightarrow$  Tree-level process with two extra particles



- **Recursive** calculation of tree-level process



# Recursive calculation in $d = 4$

Step 1

$$J_\lambda^{(6)\alpha} = ((-q - 2p) \cdot \epsilon_\lambda) J^{(2)\alpha} + (-p \cdot J^{(2)}) \epsilon_\lambda^\alpha + (J^{(2)} \cdot \epsilon_\lambda) (p - q)^\alpha$$

Step 2

$$J_\lambda^{(14)\alpha} = ((-q - 2p) \cdot \omega_\lambda) J^{(6)\alpha} + ((-q - p) \cdot J^{(6)}) \omega_\lambda^\alpha + (J^{(6)} \cdot \omega_\lambda) (p + 2q)^\alpha$$

$$\hookrightarrow \boxed{N(q) = \sum_{\lambda} \left( J_\lambda^{(14)} \cdot J^{(1)} \right)} \rightarrow \text{Numerator in } d = 4$$

# Moving to $d$ dimensions

- Obtaining the full  $d$ -dimensional expression,  $\bar{N}(\bar{q})$ , from  $N(q)$  is straightforward *analytically*. Just use the following replace rules at any step of the recursion:

$$q^2 X \rightarrow (q^2 + \mu) X \quad \sum_{\lambda} (\varepsilon_{\lambda} \cdot \omega_{\lambda}) X \rightarrow \sum_{\lambda} (\varepsilon_{\lambda} \cdot \omega_{\lambda}) X + (d - 4) X$$
$$\sum_{\lambda} (\varepsilon_{\lambda} \cdot q) (\omega_{\lambda} \cdot q) X \rightarrow \sum_{\lambda} (\varepsilon_{\lambda} \cdot q) (\omega_{\lambda} \cdot q) X + \mu X$$

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  - ∴ Numerical recursion keeps “memory” of the previous step only, not of the full recursion history!

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- This approach is simple if one uses *analytic* expressions. It is not as simple when the computation is performed *numerically*
  - ∴ Numerical recursion keeps “memory” of the previous step only, not of the full recursion history!
- We need to track the *coefficients* of all critical *Lorentz structures* which can originate  $\mu$  or  $(d - 4)$  terms during the recursion

# Moving to $d$ dimensions

- Obtaining the full  $d$ -dimensional expression,  $\bar{N}(\bar{q})$ , from  $N(q)$  is straightforward *analytically*. Just use the following replace rules at any step of the recursion:

$$\begin{aligned} q^2 X &\rightarrow (q^2 + \mu) X & \sum_{\lambda} (\varepsilon_{\lambda} \cdot \omega_{\lambda}) X &\rightarrow \sum_{\lambda} (\varepsilon_{\lambda} \cdot \omega_{\lambda}) X + (d - 4) X \\ \sum_{\lambda} (\varepsilon_{\lambda} \cdot q) (\omega_{\lambda} \cdot q) X &\rightarrow \sum_{\lambda} (\varepsilon_{\lambda} \cdot q) (\omega_{\lambda} \cdot q) X + \mu X \end{aligned}$$

- This approach is simple if one uses *analytic* expressions. It is not as simple when the computation is performed *numerically*
  - $\therefore$  Numerical recursion keeps “memory” of the previous step only, not of the full recursion history!
- We need to track the *coefficients* of all critical *Lorentz structures* which can originate  $\mu$  or  $(d - 4)$  terms during the recursion



For pure YM at 1-loop:

|                      |                                |   |  |
|----------------------|--------------------------------|---|--|
| $\tilde{q}^{\alpha}$ | $\tilde{\varepsilon}^{\alpha}$ | $(\tilde{\varepsilon} \cdot \tilde{q})$ | $(\tilde{\varepsilon} \cdot \tilde{q}) \tilde{q}^{\alpha}$ |
|----------------------|--------------------------------|---|--|

# Moving to $d$ dimensions

- Introduce formal operator  $\mathcal{E}$  which generates all *evanescent contributions* to an input  $d = 4$  current  $J^{(N)}$  at any step of the recursion:

$$\begin{aligned} \mathcal{E}[J_{\text{gluon}}^{(N)\alpha}] = & \boxed{C_q^{(N)}} \tilde{q}^\alpha + \boxed{C_\varepsilon^{(N)}} \tilde{\varepsilon}_\lambda^\alpha + \boxed{J_{q\varepsilon}^{(N)\alpha}} (\tilde{q} \cdot \tilde{\varepsilon}_\lambda) + \boxed{C_{q\varepsilon,q}^{(N)}} (\tilde{q} \cdot \tilde{\varepsilon}_\lambda) \tilde{q}^\alpha \\ & + \boxed{J_\mu^{(N)\alpha}} + \boxed{J_{d-4}^{(N)\alpha}} \end{aligned}$$

**pure YM @ 1-loop**

# Moving to $d$ dimensions

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pure YM @ 1-loop

- All *coefficients* above satisfy *recursion relations* which can be derived from the analytic structure of QCD vertices
  - Coefficients “evolve” during the recursion, in the same spirit of the standard current  $J^{(N)\alpha}$  in  $d = 4$
  - Note: coefficients are computed from pure  $d = 4$  quantities

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pure YM @ 1-loop

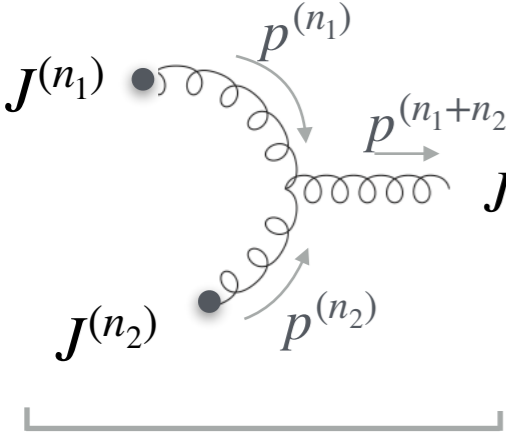
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  - Note: coefficients are computed from pure  $d = 4$  quantities

We'll show the recursion relations for the coefficients and how to combine them to obtain a numerator in  $d$  dimensions

# 1-loop recursion relations: 3-gluon vertex

[G.B, Canko, Papadopoulos and Spourdalakis, in progress]

Color-stripped vertex function:



$$\begin{aligned} J^{(n_1+n_2)} \alpha &\equiv (J^{(n_2)} \cdot (2p^{(n_1)} + p^{(n_2)})) J^{(n_1)} \alpha \\ &- (J^{(n_1)} \cdot (p^{(n_1)} + 2p^{(n_2)})) J^{(n_2)} \alpha \\ &+ (J^{(n_1)} \cdot J^{(n_2)}) (p^{(n_2)} - p^{(n_1)})^\alpha \\ &\equiv V_{3g}^\alpha (J^{(n_1)}, J^{(n_2)}) \end{aligned}$$

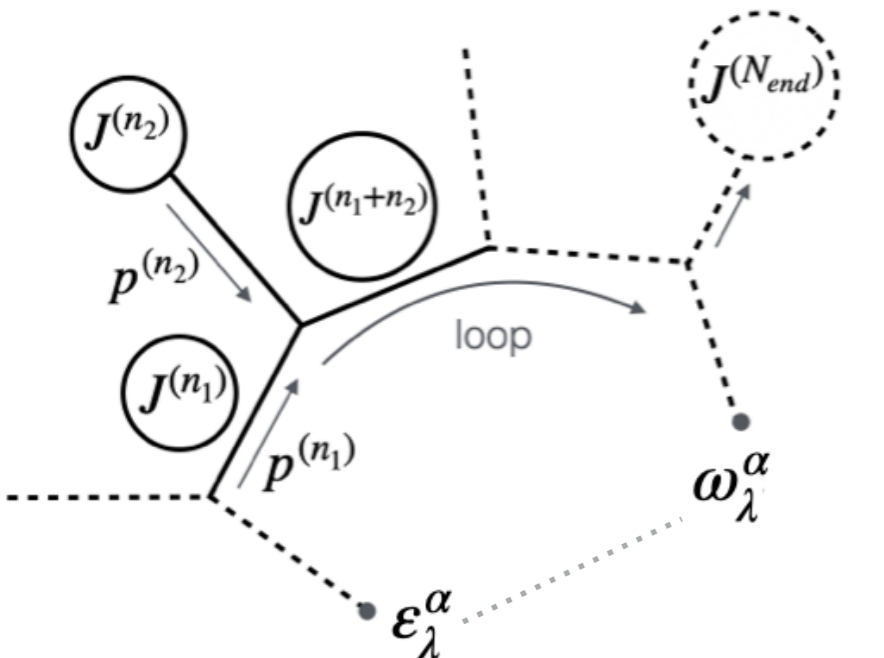


Initial conditions:

$$\begin{aligned} C_q^{(1)} &= 0 & C_\varepsilon^{(1)} &= 1 \\ C_{\varepsilon q}^{(1)} &= 0 & J_{\varepsilon q}^{(1)} \alpha &= 0 \end{aligned}$$

Example:

[loop endpoint]



$$C_q^{(n_1+n_2)} = C_q^{(n_1)} [J^{(n_2)} \cdot (2p^{(n_1)} + p^{(n_2)})] - (1 + \delta_{(n_1+n_2)N_{\text{end}}}) (J^{(n_1)} \cdot J^{(n_2)})$$

$$C_\varepsilon^{(n_1+n_2)} = C_\varepsilon^{(n_1)} [J^{(n_2)} \cdot (2p^{(n_1)} + p^{(n_2)})]$$

$$J_{\varepsilon q}^{(n_1+n_2)} \alpha = V_{3g}^\alpha (J_{\varepsilon q}^{(n_1)}, J^{(n_2)}) - (C_\varepsilon^{(n_1)} + \mu C_{\varepsilon q, q}^{(n_1)}) J^{(n_2)} \alpha$$

$$C_{\varepsilon q, q}^{(n_1+n_2)} = C_{\varepsilon q, q}^{(n_1)} [J^{(n_2)} \cdot (2p^{(n_1)} + p^{(n_2)})] - (1 + \delta_{(n_1+n_2)N_{\text{end}}}) (J_{q\varepsilon}^{(n_1)} \cdot J^{(n_2)})$$

$$J_\mu^{(n_1+n_2)} \alpha = \mu \left[ (-1 + 2 \delta_{(n_1+n_2)N_{\text{end}}} C_q^{(n_1)} J^{(n_2)} \alpha) + \delta_{(n_1+n_2)N_{\text{end}}} (J_{\varepsilon q}^{(n_1)} \alpha + C_{\varepsilon q, q}^{(n_1)} (p^{(n_2)} - p^{(n_1)})^\alpha) \right]$$

$$J_{d-4}^{(n_1+n_2)} \alpha = (d-4) \left[ \delta_{(n_1+n_2)N_{\text{end}}} C_\varepsilon^{(n_1)} (p^{(n_2)} - p^{(n_1)})^\alpha \right]$$



# 1-loop recursion relations: gluon-ghost vertices

[G.B, Canko, Papadopoulos and Spourdalakis, in progress]

Color-stripped vertex functions:

$$J^{(n_1+n_2)} \alpha \equiv - J^{(n_1)} J^{(n_2)} p^{(n_1)} \alpha$$

$$J^{(n_1+n_2)} \alpha \equiv (J^{(n_1)} \cdot (p^{(n_1)} + p^{(n_2)})) J^{(n_2)}$$

$$J^{(n_1+n_2)} \alpha \equiv (J^{(n_2)} \cdot (p^{(n_1)})) J^{(n_1)} \alpha$$



$$\mathcal{E}[J_{\text{ghost}}^{(N)}] = \cancel{J_{\mu}^{(N)}}^{=0} + \cancel{J_{d-4}^{(N)}}^{=0}$$

**QCD @ 1-loop**

No evanescent terms are generated by 1-loop gluon-ghost vertices!

∴ The gluon is never associated to a loop propagator

# Steps to full QCD @ 1-loop

Evanescent terms for *fermionic* currents

$$\begin{aligned}\mathcal{E}[J_{\text{fermion}}^{(N)}] = & \not{\tilde{q}} \psi_q^{(N)} + \not{\tilde{q}} \psi_{q,R}^{(N)} + \not{\tilde{\epsilon}}_\lambda \psi_\epsilon^{(N)} \\ & + (\tilde{q} \cdot \tilde{\epsilon}_\lambda) \psi_{q\epsilon}^{(N)} + (\tilde{q} \cdot \tilde{\epsilon}_\lambda) \not{\tilde{q}} \psi_{q\epsilon,q}^{(N)} + \tilde{j}_R^{(N)} \\ & + J_\mu^{(N)} + J_{d-4}^{(N)}\end{aligned}$$

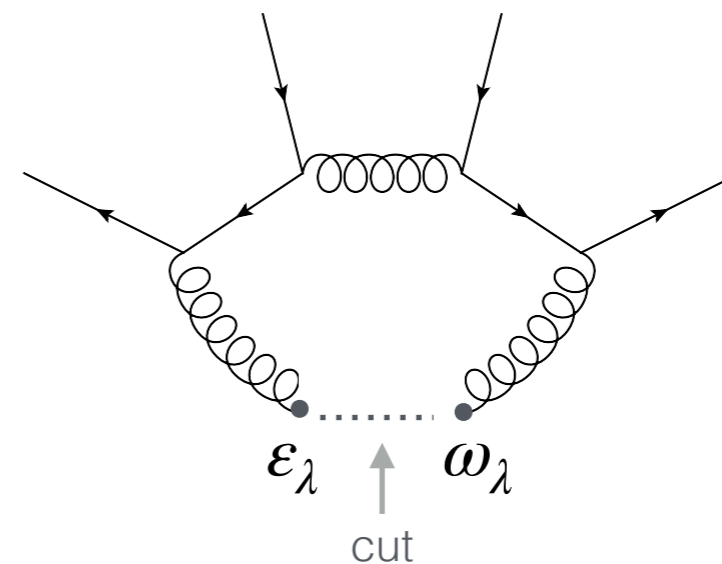
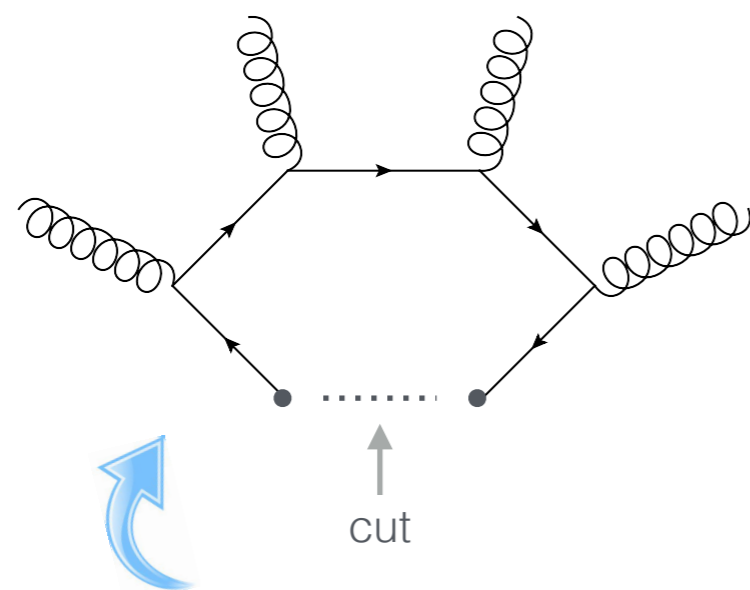
**QCD @ 1-loop**

# Steps to full QCD @ 1-loop

## Evanescent terms for fermionic currents

$$\begin{aligned}
 \mathcal{E}[J_{\text{fermion}}^{(N)}] = & \not{\tilde{q}} \boxed{\psi_q^{(N)}} + \not{\tilde{q}} \boxed{\psi_{q,R}^{(N)}} + \not{\tilde{\varepsilon}}_\lambda \boxed{\psi_\varepsilon^{(N)}} \\
 & + (\tilde{q} \cdot \tilde{\varepsilon}_\lambda) \boxed{\psi_{q\varepsilon}^{(N)}} + (\tilde{q} \cdot \tilde{\varepsilon}_\lambda) \not{\tilde{q}} \boxed{\psi_{q\varepsilon,q}^{(N)}} + \boxed{\tilde{j}_R^{(N)}} \\
 & + \boxed{J_\mu^{(N)}} + \boxed{J_{d-4}^{(N)}}
 \end{aligned}$$

QCD @ 1-loop

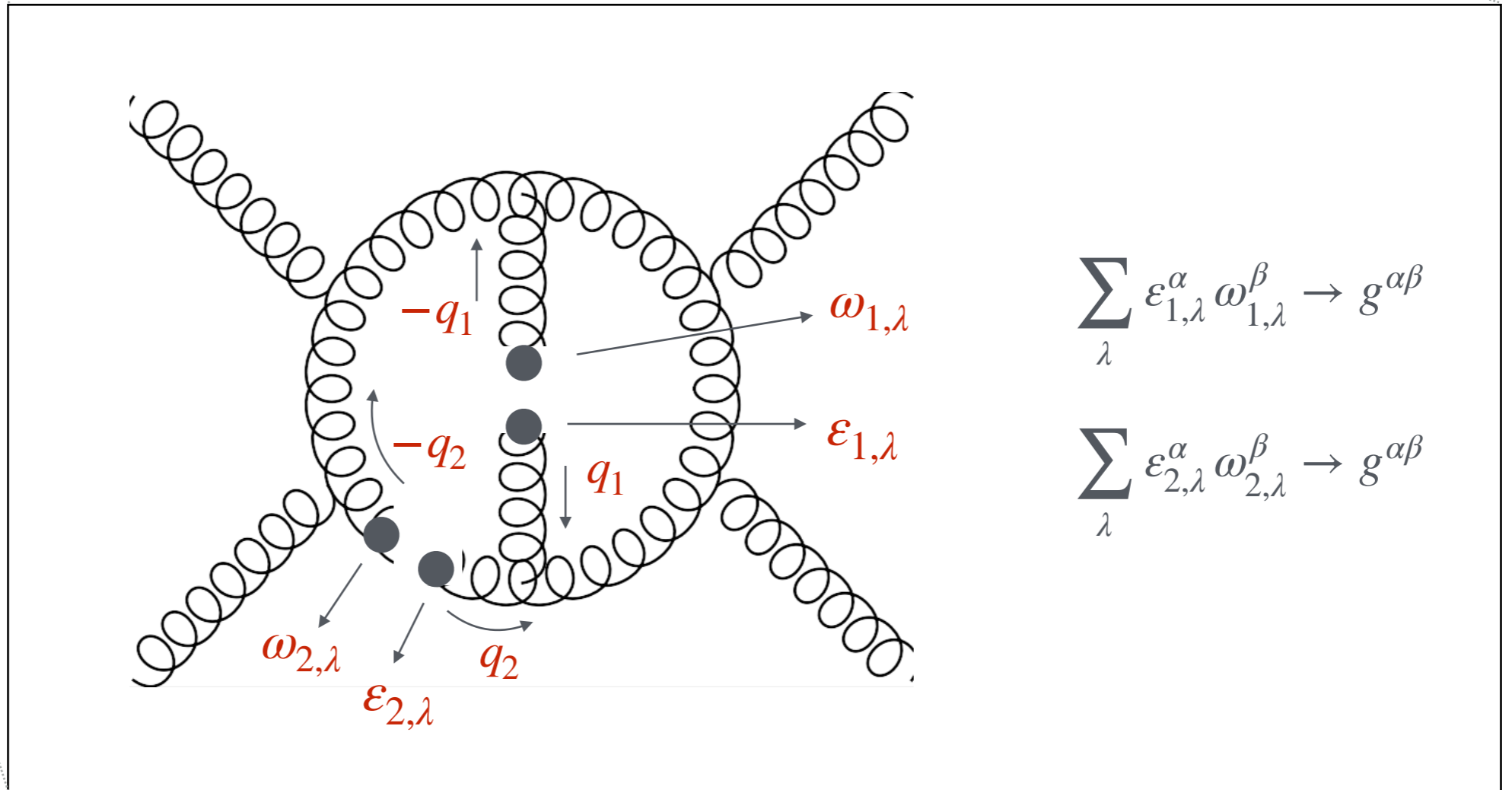
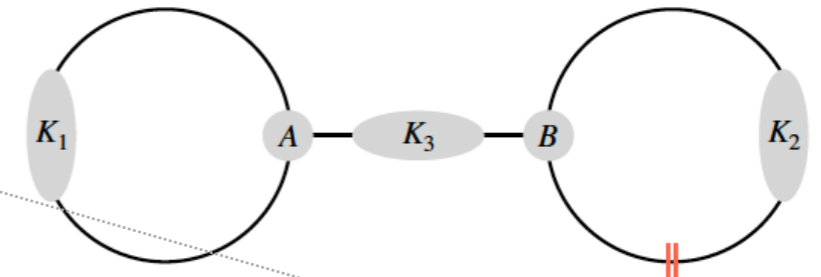
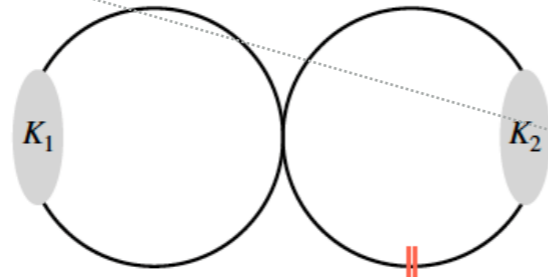
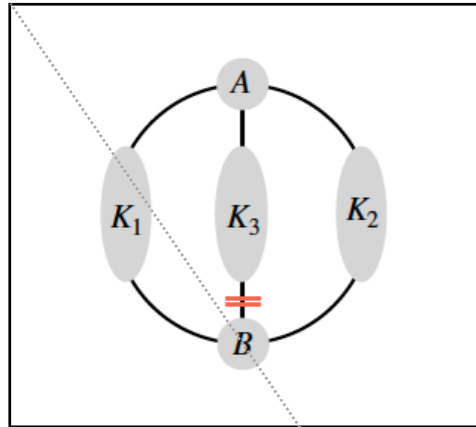


Checked for *closed-fermion loop* topologies so far. General case under study

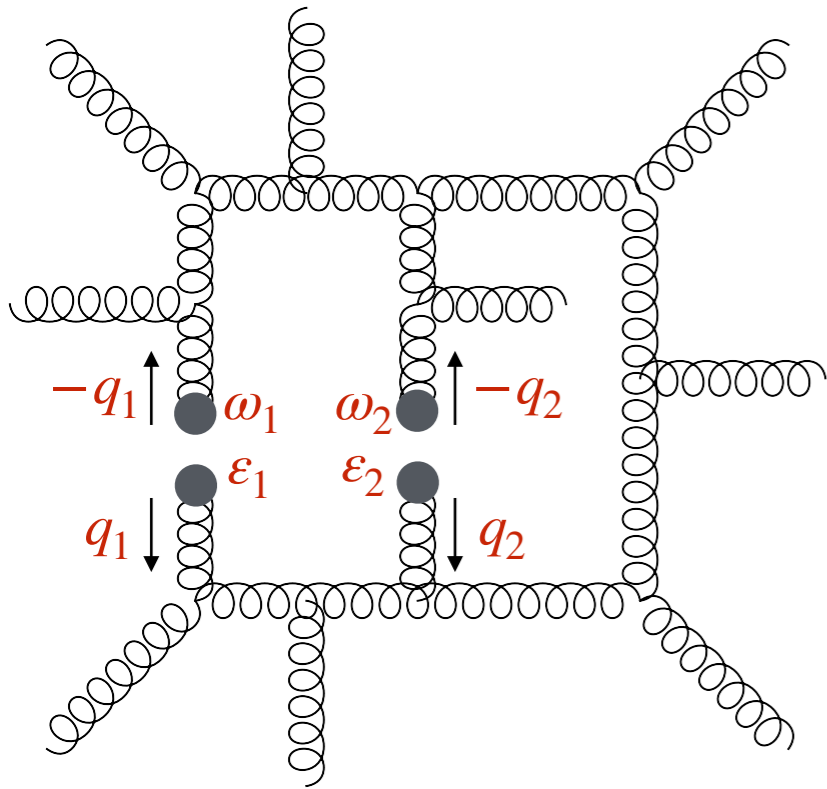
## II. Beyond 1-loop

# **Steps to 2-loop construction**

# Basic notation at 2-loop



# Extension to the 2-loop case



Origin of evanescent terms at 2-loops:

$$\mathcal{E}[(q_i \cdot q_j) X] = \mu_{ij} X \quad [i, j = 1, 2]$$

$$\mathcal{E}\left[\sum_{\lambda} (q_i \cdot \varepsilon_{k, \lambda}) (q_j \cdot \omega_{k, \lambda}) X\right] = \mu_{ij} X \quad [i, j = 1, 2]$$

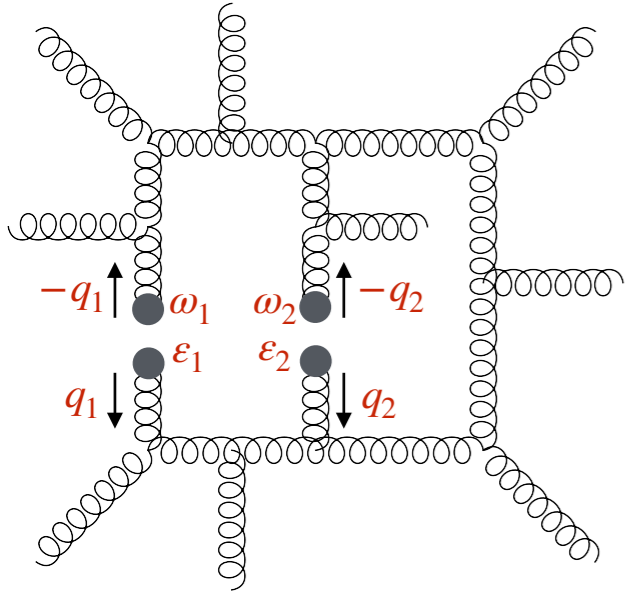
$$\mathcal{E}\left[\sum_{\lambda} (\varepsilon_{i, \lambda} \cdot \omega_{i, \lambda}) X\right] = (d - 4) X \quad [i, j = 1, 2]$$

$$\mathcal{E}\left[\sum_{\lambda_1, \lambda_2} (\varepsilon_{1, \lambda_1} \cdot \omega_{1, \lambda_1}) (\varepsilon_{2, \lambda_2} \cdot \omega_{2, \lambda_2}) X\right] = (d - 4) X$$

$$\mathcal{E}\left[\sum_{\lambda_1, \lambda_2} (\varepsilon_{1, \lambda_1} \cdot \omega_{2, \lambda_2}) (\varepsilon_{2, \lambda_2} \cdot \omega_{1, \lambda_1}) X\right] = (d - 4) X$$

$$\mathcal{E}\left[\sum_{\lambda_1, \lambda_2} (\varepsilon_{1, \lambda_1} \cdot \varepsilon_{2, \lambda_2}) (\omega_{1, \lambda_1} \cdot \omega_{2, \lambda_2}) X\right] = (d - 4) X$$

# Extension to the 2-loop case



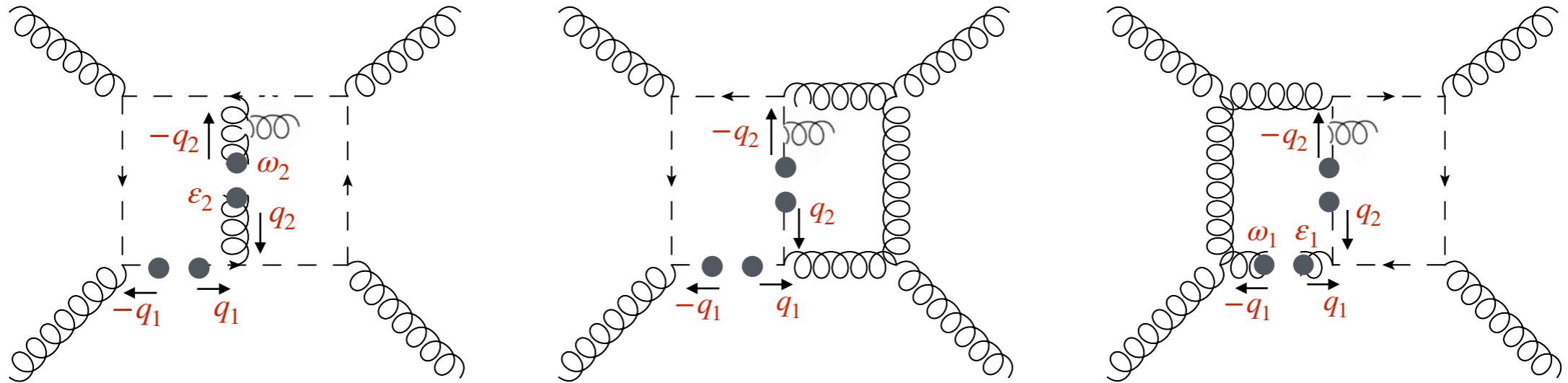
At two loops, more critical structures ( $\Rightarrow$  coefficients) need to be tracked during the recursion



$$\begin{aligned}
 \mathcal{E}[J_{\text{gluon}}^{(N)\alpha}] \equiv & \sum_{i=1}^2 C_{q_i}^{(N)} \tilde{q}_i^\alpha + \sum_{i=1}^2 C_{\varepsilon_i}^{(N)} \tilde{\varepsilon}_i^\alpha + C_{\omega_2}^{(N)} \tilde{\omega}_2^\alpha \\
 & + \sum_{i,j=1}^2 J_{\varepsilon_i q_j}^{(N)\alpha} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J_{\omega_2 q_2}^{(N)\alpha} (\tilde{\omega}_2 \cdot \tilde{q}_2) + J_{\varepsilon_1 \varepsilon_2}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^2 J_{\varepsilon_1 q_i \varepsilon_2 q_j}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
 & + \sum_{i,j,k=1}^2 C_{\varepsilon_i q_j q_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{q}_k^\alpha + C_{\omega_2 q_2 q_2}^{(N)} (\tilde{\omega}_2 \cdot \tilde{q}_2) \tilde{q}_2^\alpha + \sum_{\substack{i,j,k=1 \\ k \neq i}}^2 C_{\varepsilon_i q_j \varepsilon_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{\varepsilon}_k^\alpha \\
 & + \sum_{i,j=1}^2 C_{\varepsilon_1 \varepsilon_2 q_i}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) \tilde{q}_i^\alpha + \sum_{i,j,k=1}^2 C_{\varepsilon_1 q_i \varepsilon_2 q_j q_k}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i) (\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \tilde{q}_k^\alpha \\
 & + J_{\mu}^{(N)\alpha} + J_{(d-4)}^{(N)\alpha}
 \end{aligned}$$

pure YM @ 2-loop

# Extension to the 2-loop case



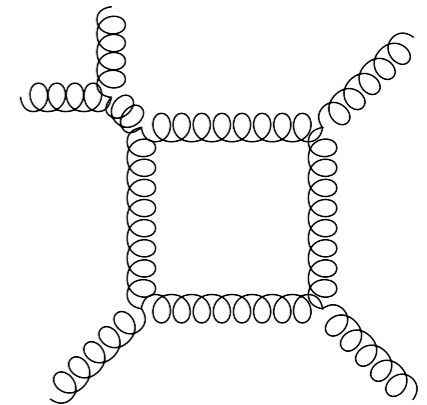
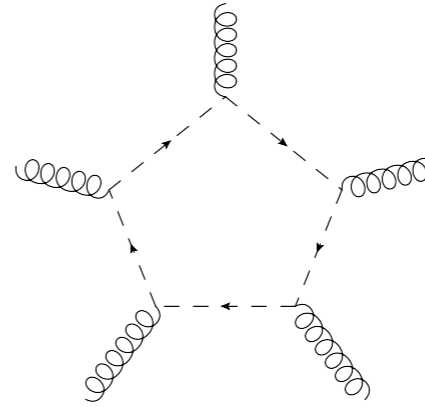
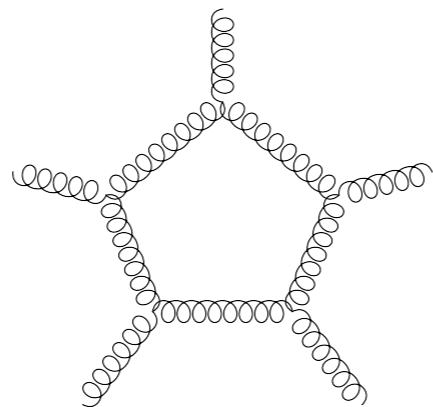
$$\mathcal{E}[J_{\text{ghost}}^{(N)}] \equiv \sum_{i,j=1}^2 C_{\varepsilon_i q_j}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J_{\mu}^{(N)} \alpha + \cancel{J_{(d-4)}^{(N)} \alpha} = 0$$

**pure YM @ 2-loop**

Starting from 2-loops, also ghost vertices develop  $\mu_{ij}$  terms through the recursion

## I. Completed numerical validation for pure YM in HELAC-ILOOP:

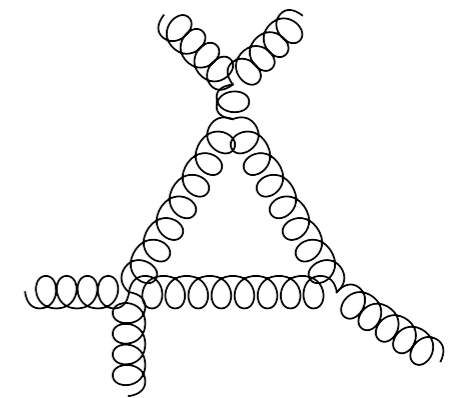
- $gg \rightarrow gg$
- $gg \rightarrow ggg$



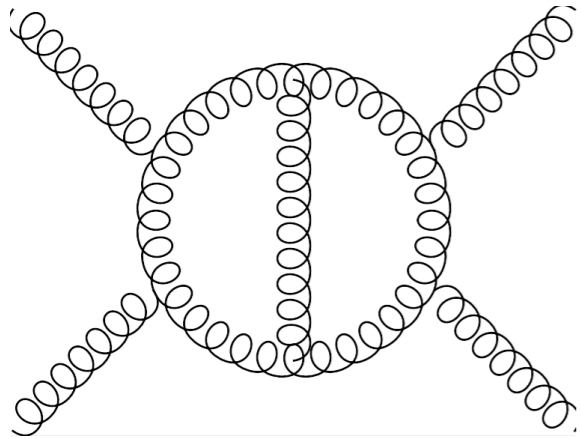
- Cut Constructible (CC) + Rational Terms computed at *integrand* level



- $CC+R_1$  at integrand level,  $R_2$  via effective Feynman rules (standard approach)



## II. Analytic proof of concept of recursion @ 2-loop $\rightarrow$ double-box topology



$\mu_{ij}$  and  $(d-4)$  terms generated recursively

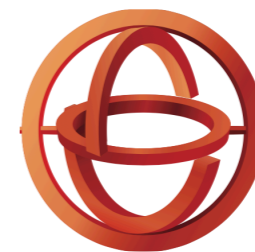
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 & + J_{\mu}^{(N)\alpha} + J_{(d-4)}^{(N)\alpha}
 \end{aligned}$$

Numerical implementation in HELAC-2LOOP in progress

- Work is underway to enable HELAC framework to perform numerical computations of loop numerators in  $d = 4 - 2\varepsilon$  dimensions
- Interesting applications to integrand-level reduction of 2-loop amplitudes
- We are formulating a new method for computing  $\mu_{ij}$  and  $(d - 4)$  contributions in the context of *recursive* numerator computations
- The method has been established for pure YM at 2-loops
- Extension to quarks and numerical implementation in HELAC-2LOOP in progress

Thank you for your attention!

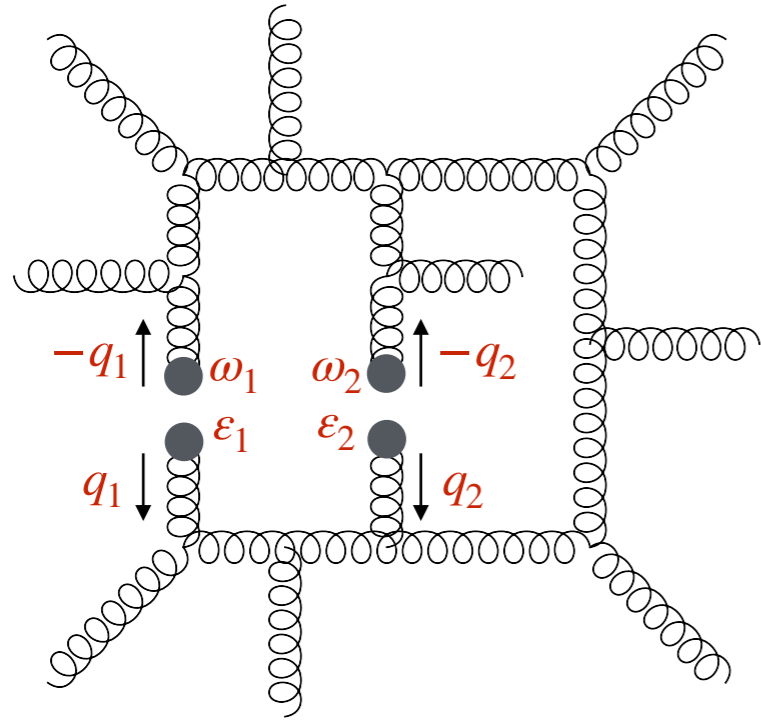
This research is supported by the Hellenic foundation for Research and Innovation (H.F.R.I.) under the “*2nd Call for H.F.R.I. Research Projects to support Faculty Members and Researchers*” (Project Number: 02674 - HOCTools-II)



**H.F.R.I.**  
Hellenic Foundation for  
Research & Innovation

# **Backup slides**

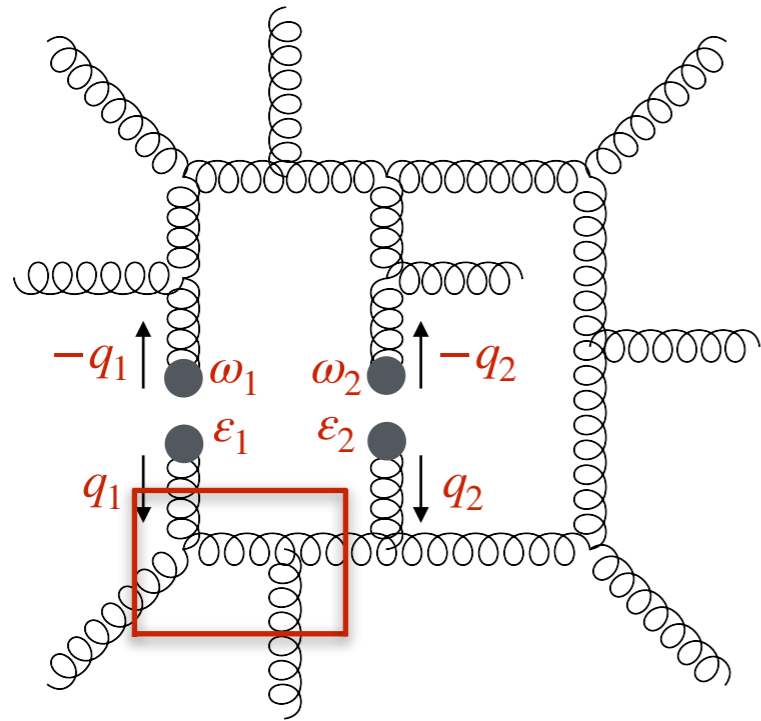
# Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$\begin{aligned}
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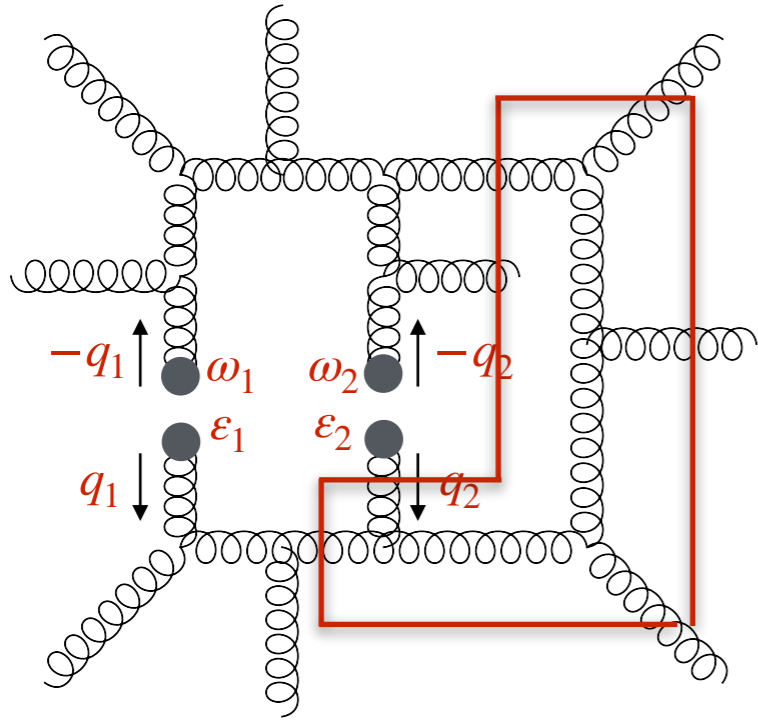
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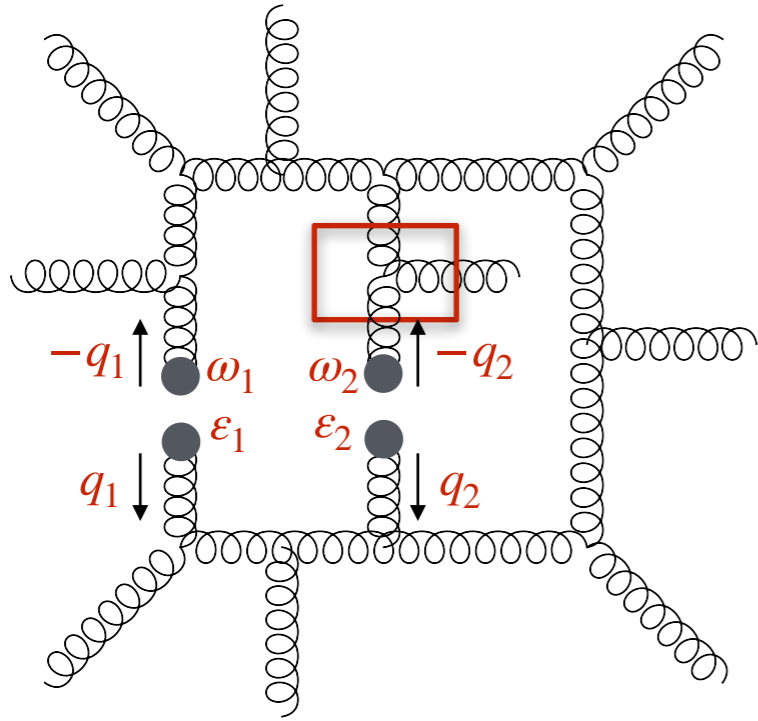
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 \mathcal{E}[J_{\text{gluon}}^{(N)\alpha}] \equiv & \sum_{i=1}^2 C_{q_i}^{(N)} \tilde{q}_i^\alpha + \sum_{i=1}^2 C_{\varepsilon_i}^{(N)} \tilde{\varepsilon}_i^\alpha + C_{\omega_2}^{(N)} \tilde{\omega}_2^\alpha \\
 & + \sum_{i,j=1}^2 J_{\varepsilon_i q_j}^{(N)\alpha} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J_{\omega_2 q_2}^{(N)\alpha} (\tilde{\omega}_2 \cdot \tilde{q}_2) + J_{\varepsilon_1 \varepsilon_2}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^2 J_{\varepsilon_1 q_i \varepsilon_2 q_j}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i) (\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
 & + \sum_{i,j,k=1}^2 C_{\varepsilon_i q_j q_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{q}_k^\alpha + C_{\omega_2 q_2 q_2}^{(N)} (\tilde{\omega}_2 \cdot \tilde{q}_2) \tilde{q}_2^\alpha + \sum_{\substack{i,j,k=1 \\ k \neq i}}^2 C_{\varepsilon_i q_j \varepsilon_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{\varepsilon}_k^\alpha \\
 & + \sum_{i,j=1}^2 C_{\varepsilon_1 \varepsilon_2 q_i}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) \tilde{q}_i^\alpha + \sum_{i,j,k=1}^2 C_{\varepsilon_1 q_i \varepsilon_2 q_j q_k}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i) (\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \tilde{q}_k^\alpha \\
 & + J_{\mu}^{(N)\alpha} + J_{(d-4)}^{(N)\alpha}
 \end{aligned}$$

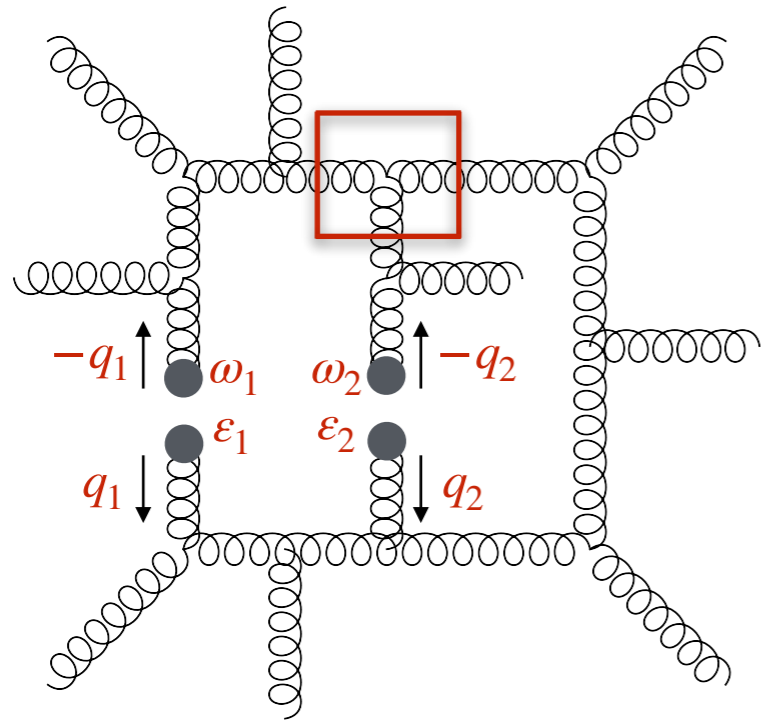
# Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$\begin{aligned}
 \mathcal{E}[J_{\text{gluon}}^{(N)\alpha}] \equiv & \sum_{i=2}^2 C_{q_i}^{(N)} \tilde{q}_i^\alpha + \sum_{i,j=1}^2 C_{\varepsilon_i}^{(N)} \tilde{\varepsilon}_i^\alpha + C_{\omega_2}^{(N)} \tilde{\omega}_2^\alpha \\
 & + \sum_{i,j=1}^2 J_{\varepsilon_i q_j}^{(N)\alpha} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J_{\omega_2 q_2}^{(N)\alpha} (\tilde{\omega}_2 \cdot \tilde{q}_2) + J_{\varepsilon_1 \varepsilon_2}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^2 J_{\varepsilon_1 q_i \varepsilon_2 q_j}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i) (\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
 & + \sum_{i,j,k=1}^2 C_{\varepsilon_i q_j q_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{q}_k^\alpha + C_{\omega_2 q_2 q_2}^{(N)} (\tilde{\omega}_2 \cdot \tilde{q}_2) \tilde{q}_2^\alpha + \sum_{\substack{i,j,k=1 \\ k \neq i}}^2 C_{\varepsilon_i q_j \varepsilon_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{\varepsilon}_k^\alpha \\
 & + \sum_{i,j=1}^2 C_{\varepsilon_1 \varepsilon_2 q_i}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) \tilde{q}_i^\alpha + \sum_{i,j,k=1}^2 C_{\varepsilon_1 q_i \varepsilon_2 q_j q_k}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i) (\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \tilde{q}_k^\alpha \\
 & + J_{\mu}^{(N)\alpha} + J_{(d-4)}^{(N)\alpha}
 \end{aligned}$$

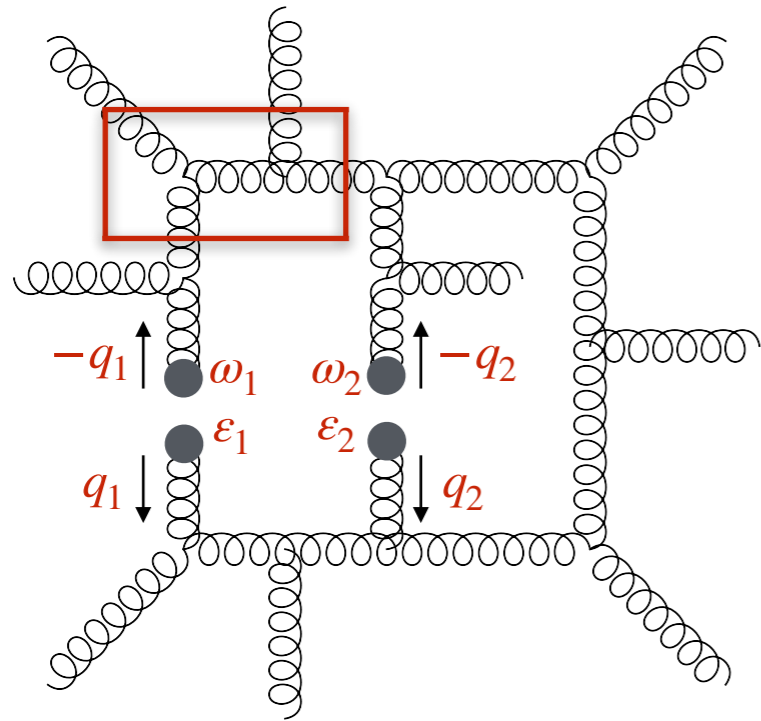
# Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$\begin{aligned}
 \mathcal{E}[J_{\text{gluon}}^{(N)\alpha}] \equiv & \sum_{i=1}^2 C_{q_i}^{(N)} \tilde{q}_i^\alpha + \sum_{i=1}^1 C_{\epsilon_i}^{(N)} \tilde{\epsilon}_i^\alpha + C_{\omega_2}^{(N)} \tilde{\omega}_2^\alpha \\
 & + \sum_{i,j=1}^2 J_{\epsilon_i q_j}^{(N)\alpha} (\tilde{\epsilon}_i \cdot \tilde{q}_j) + J_{\omega_2 q_2}^{(N)\alpha} (\tilde{\omega}_2 \cdot \tilde{q}_2) + J_{\epsilon_1 \epsilon_2}^{(N)\alpha} (\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2) + \sum_{i,j=1}^2 J_{\epsilon_1 q_i \epsilon_2 q_j}^{(N)\alpha} (\tilde{\epsilon}_1 \cdot \tilde{q}_i)(\tilde{\epsilon}_2 \cdot \tilde{q}_j) \\
 & + \sum_{i,j,k=1}^2 C_{\epsilon_i q_j q_k}^{(N)} (\tilde{\epsilon}_i \cdot \tilde{q}_j) \tilde{q}_k^\alpha + C_{\omega_2 q_2 q_2}^{(N)} (\tilde{\omega}_2 \cdot \tilde{q}_2) \tilde{q}_2^\alpha + \sum_{\substack{i,j,k=1 \\ k \neq i}}^2 C_{\epsilon_i q_j \epsilon_k}^{(N)} (\tilde{\epsilon}_i \cdot \tilde{q}_j) \tilde{\epsilon}_k^\alpha \\
 & + \sum_{i,j=1}^2 C_{\epsilon_1 \epsilon_2 q_i}^{(N)} (\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2) \tilde{q}_i^\alpha + \sum_{i,j,k=1}^2 C_{\epsilon_1 q_i \epsilon_2 q_j q_k}^{(N)} (\tilde{\epsilon}_1 \cdot \tilde{q}_i)(\tilde{\epsilon}_2 \cdot \tilde{q}_j) \tilde{q}_k^\alpha \\
 & + J_\mu^{(N)\alpha} + \boxed{J_{(d-4)}^{(N)\alpha}}
 \end{aligned}$$

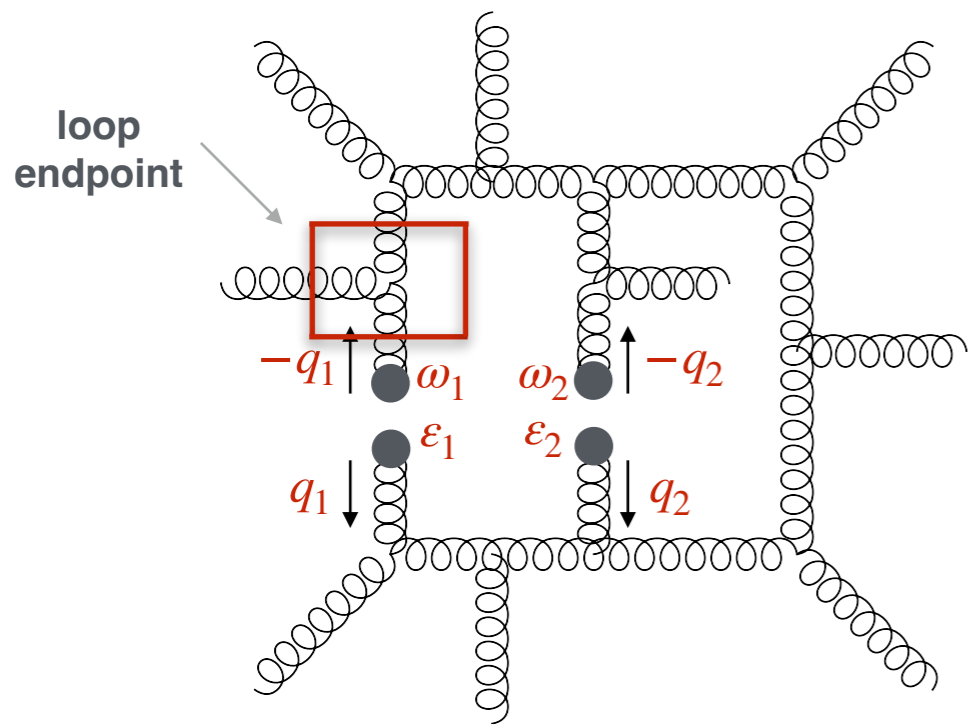
# Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$\begin{aligned}
 \mathcal{E}[J_{\text{gluon}}^{(N)\alpha}] \equiv & \sum_{i=1}^2 C_{q_i}^{(N)} \tilde{q}_i^\alpha + \sum_{i=1}^1 C_{\varepsilon_i}^{(N)} \tilde{\varepsilon}_i^\alpha + C_{\omega_2}^{(N)} \tilde{\omega}_2^\alpha \\
 & + \sum_{j=1}^2 J_{\varepsilon_i q_j}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{q}_j) + J_{\omega_2 q_2}^{(N)\alpha} (\tilde{\omega}_2 \cdot \tilde{q}_2) + J_{\varepsilon_1 \varepsilon_2}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^2 J_{\varepsilon_1 q_i \varepsilon_2 q_j}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i) (\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
 & + \sum_{j,k=1}^2 C_{\varepsilon_1 q_j q_k}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{q}_j) \tilde{q}_k^\alpha + C_{\omega_2 q_2 q_2}^{(N)} (\tilde{\omega}_2 \cdot \tilde{q}_2) \tilde{q}_2^\alpha + \sum_{\substack{i,j,k=1 \\ k \neq i}}^2 C_{\varepsilon_i q_j \varepsilon_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{\varepsilon}_k^\alpha \\
 & + \sum_{i,j=1}^2 C_{\varepsilon_1 \varepsilon_2 q_i}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) \tilde{q}_i^\alpha + \sum_{i,j,k=1}^2 C_{\varepsilon_1 q_i \varepsilon_2 q_j q_k}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i) (\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \tilde{q}_k^\alpha \\
 & + J_{\mu}^{(N)\alpha} + J_{(d-4)}^{(N)\alpha}
 \end{aligned}$$

# Strategy of calculation



Not *all* coefficients are involved at each step of the recursion!

$$\begin{aligned}
 \mathcal{E}[J_{\text{gluon}}^{(N)\alpha}] \equiv & \sum_{i,j=1}^2 C_{q_i}^{(N)} \tilde{q}_i^\alpha + \sum_{i,j=1}^2 C_{\varepsilon_i}^{(N)} \tilde{\varepsilon}_i^\alpha + C_{\omega_2}^{(N)} \tilde{\omega}_2^\alpha \\
 & + \sum_{i,j=1}^2 J_{\varepsilon_i q_j}^{(N)\alpha} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) + J_{\omega_2 q_2}^{(N)\alpha} (\tilde{\omega}_2 \cdot \tilde{q}_2) + J_{\varepsilon_1 \varepsilon_2}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) + \sum_{i,j=1}^2 J_{\varepsilon_1 q_i \varepsilon_2 q_j}^{(N)\alpha} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \\
 & + \sum_{i,j,k=1}^2 C_{\varepsilon_i q_j q_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{q}_k^\alpha + C_{\omega_2 q_2 q_2}^{(N)} (\tilde{\omega}_2 \cdot \tilde{q}_2) \tilde{q}_2^\alpha + \sum_{\substack{i,j,k=1 \\ k \neq i}}^2 C_{\varepsilon_i q_j \varepsilon_k}^{(N)} (\tilde{\varepsilon}_i \cdot \tilde{q}_j) \tilde{\varepsilon}_k^\alpha \\
 & + \sum_{i,j=1}^2 C_{\varepsilon_1 \varepsilon_2 q_i}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2) \tilde{q}_i^\alpha + \sum_{i,j,k=1}^2 C_{\varepsilon_1 q_i \varepsilon_2 q_j q_k}^{(N)} (\tilde{\varepsilon}_1 \cdot \tilde{q}_i)(\tilde{\varepsilon}_2 \cdot \tilde{q}_j) \tilde{q}_k^\alpha \\
 & + J_\mu^{(N)\alpha} + \boxed{J_{(d-4)}^{(N)\alpha}}
 \end{aligned}$$