Construction of two-loop amplitudes with HELAC

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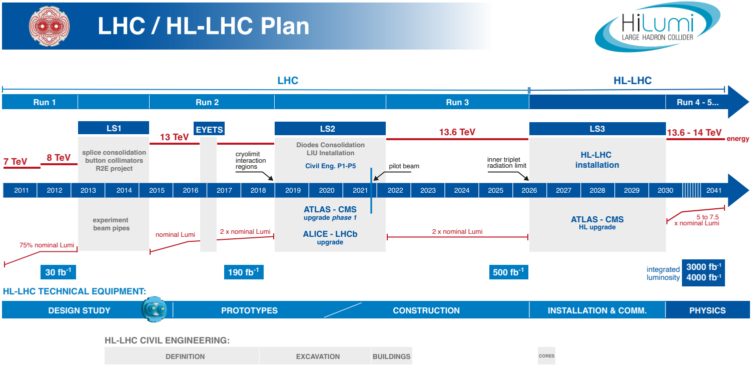
Introduction



Future 14 TeV HL-LHC runs

⇓

Observables measured with percent accuracy

⇓

NNLO or N3LO corrections demanded from theory side

Cross Section Computations in Perturbative QCD

- QCD amplitudes expanded in $\alpha_s \to$ X-loop contributions to hard-scattering cross section @ NXLO

$$d\hat{\sigma}^{NXLO}_{ij\to f} \sim \int d\Phi^{(n-2)} \left(2\,\mathrm{Re}\left[\mathcal{M}_n^{(X)} (\mathcal{M}_n^{(0)})^* \right] \right) + \dots$$

- $\mathcal{M}_n^{(X)}$ is a sum of X-loop Feynman Diagrams $\to$ Feynman Integrals (FIs) $[d = 4 - 2\epsilon]$

$$\mathcal{M}_n^{(X)} = \sum_t C_t F_t \quad \text{with} \quad F_t = \int \mathrm{d}^d k_1 \dots \mathrm{d}^d k_X \prod_{i\in t} \frac{1}{D_i^{a_i}}\,, \quad D_i = (k_i + p_i)^2 - m_i^2$$

which satisfy integration-by-part relations [Chetyrkin & Tkachov, 1981; Laporta, 2004] $\to$ Master Integrals

$$\int \mathrm{d}^d k_1 \dots \mathrm{d}^d k_X \frac{\partial}{\partial k_j^\mu} \prod_{i\in t} \frac{1}{D_i^{a_i}} = 0 \quad \longrightarrow \quad \{\mathcal{J}_i\} \quad \text{(See Gaia's Talk)}$$

- $\mathcal{M}_n^{(X)}$ expressed as a sum of Master Integrals (MIs) multiplied by process-dependent coefficients

$$\boxed{\mathcal{M}_n^{(X)} = \sum_{i=1}^{N_{\mathrm{MIs}}} \tilde{C}_i \mathcal{J}_i}$$

(See talks of Michal and Matteo for recent MI computations)

Status of NNLO QCD predictions

- Current frontier of NNLO QCD predictions stands at $2 \to 3$ processes [Les Houches 2023]

$$pp \to H + 2j,\ H/V'/j + t\bar{t},\ V + b\bar{b},\ VV' + j,\ tZj \text{ with } V' = V, \gamma \text{ and } V = Z, W$$

- Recent cross-section results

Process	Color	Reference
$pp \to \gamma\gamma\gamma$	LC	Kallweit, Sotnikov & Wiesemann, 2021
$pp \to \gamma\gamma j$	LC	Chawdhry, Czakon, Mitov & Poncelet, 2021
$pp \to jjj$	LC	Czakon, Mitov & Poncelet, 2021
$pp \to b\bar{b}W$	LC	Hartanto, Poncelet, Popescu & Zoia, 2022
$pp \to b\bar{b}Z$	LC	Mazzitelli, Sotnikov & Wiesemann, 2024
$pp \to \gamma\gamma + j$	FC	Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet & Zoia, 2023
$pp \to \gamma\gamma j$	FC	Buccioni, Chen, Feng, Gehrmann, Huss & Marcoli, 2025
$q\bar{q} \to \gamma\gamma\gamma$	FC	Kermanschah & Vicini (2025)

Apologies if I missed any relevant citations or results herein and from here on!

Recent Results for $2 \to 3$ *Two-Loop Amplitudes*

Process	Color	Reference
$pp \to Vjj$	LC	De Laurentis, Ita, Page & Vasily Sotnikov (2025)
$gg \to t\bar{t}g$	LC	Badger, Becchetti, Brancaccio, Hartanto & Zoia (2025)
$pp \to W\gamma j$	LC	Badger, Hartanto, Kryś & Zoia (2022)
$pp \to b\bar{b}H$	LC	Badger, Hartanto, Kryś & Zoia (2021)
$u\bar{d} \to b\bar{b}W^+$	LC	Badger, Hartanto & Zoia (2021)
$W + 4$ partons	LC	Badger, Hartanto, Hansen & Peraro (2019), Abreu, Cordero, Ita, Klinkert, Page & Sotnikov (2022)
$gg/q\bar{q} \to g\gamma\gamma$	LC	Badger, Chicherin, Gehrmann, Hartanto, Hansen, Henn, Marcoli, Moodie, Peraro & Zoia (2021)
$q\bar{q} \to \gamma\gamma\gamma$	LC	Chawdhry, Czakon, Mitov & Poncelet (2021), Abreu, Page, Pascual & Sotnikov (2021)
$pp \to 3j$	LC	Abreu, Cordero, Ita, Page, & Sotnikov (2021)
5 partons	LC	Abreu, Dormans, Cordero, Ita, Page & Sotnikov (2019), Abreu, Cordero, Ita, Page & Sotnikov, (2018)
$gg \to ggg$	LC	Abreu, Dormans, Cordero, Ita, Page & Zeng (2018 & 2019), Badger, Hansen, Hartanto & Peraro (2019)
$pp \to b\bar{b}H$	FC	Badger, Hartanto, Poncelet, Wu, Zhang & Zoia (2025)
$u\bar{d} \to \gamma\gamma W^+$	FC	Badger, Hartanto, Wu, Zhang & Zoia (2024)
5 partons	FC	Agarwal, Buccioni, Devoto, Gambuti, Manteuffel, Tancredi, De Laurentis, Ita, Klinkert & Sotnikov (2023)
$q\bar{q} \to \gamma\gamma\gamma$	FC	Abreu, De Laurentis, Ita, Klinkert, Page & Sotnikov (2023)
$q\bar{q} \to \gamma gg/Q\bar{Q}$	FC	Badger, Czakon, Hartanto, Moodie, Peraro, Poncelet & Zoia (2023)
$q\bar{q} \to g\gamma\gamma$	FC	Agarwal, Buccioni, Manteuffel & Tancredi (2021)
$gg \to ggg$	FC	Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang & Zoia (2019)

Color Decomposition of QCD Amplitude

- In QCD, $\mathcal{M}_n^{(L)}$ can be decomposed into a color factor and a kinematic-dependent part

$$\mathcal{M}_n^{(L)} = \sum_F c_F^{(L)} \mathcal{A}_{n,F}^{(L)}.$$

- Several color representations (bases) $\rightarrow$ most famous the fundamental one $\rightarrow$ $c_F^{(L)}$ expressed in terms of traces of the $SU(3)$ generators (t_{ij}^a).

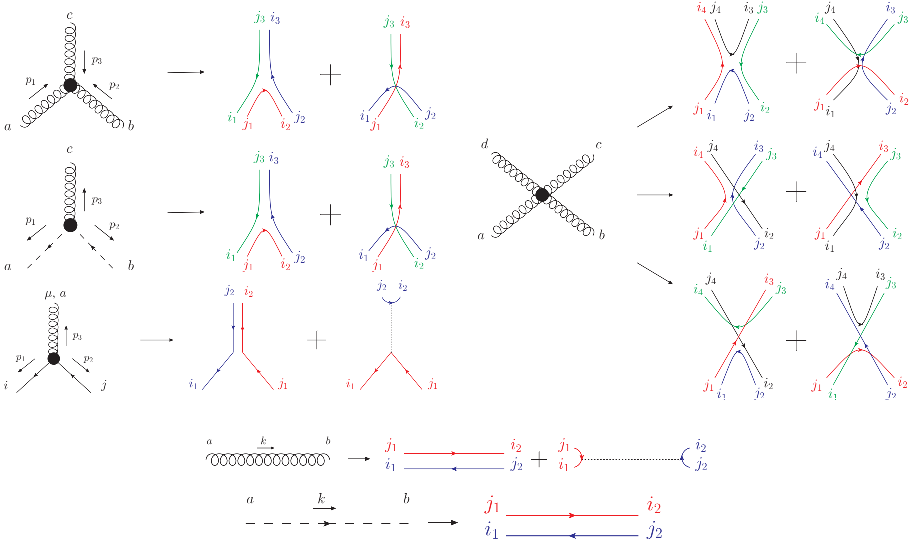
- We use the color-flow basis $\rightarrow$ $\mathcal{M}_n^{(L)}$ contracted with a t_{ij}^a (in the adjoint index) for every gluon

$$\mathcal{M}_n^{(L)} = \sum_{\sigma}^{k!} \delta_{j_1}^{i_{\sigma_1}} \delta_{j_2}^{i_{\sigma_2}} \cdots \delta_{j_k}^{i_{\sigma_k}} \mathcal{A}_{n,\sigma}^{(L)}, \quad \text{with} \quad k = n_g + n_{q\bar{q}}.$$

- In this representation:

1. Gluons and ghosts are represented by a pair of color/anti-color indices/lines (i, j).
2. Quarks (anti-quarks) are represented by a single color $(i, 0)$ (anti-color $(0, j)$) index.
3. All the rest of the particles (not carry color) have $(0, 0)$.

Color-Flow Feynman Rules



HELAC-2LOOP under construction

- This Talk → Construction of two-loop scattering amplitudes (skeletons)

$$\mathcal{M}_n^{(L)} = \sum_F c_F^{(L)} \int \left(\prod_{i=1}^{L} \frac{d^d k_i}{(2\pi)^d} \mu^{4-d} \right) \sum_{I \subseteq T} \frac{N_{I,F}\left(\{k\},\{p\},\gamma^\mu,\{\epsilon^\mu,u,\upsilon\}\right)}{\prod_{j\in I} D_j^{a_j}(\{k\},\{p\},m_j)}$$

using a Hybrid approach: Topology generation + Dyson-Schwinger recursion!

- Numerical evaluation en-capturing the $(d-4)$-dependency of the amplitude → Giuseppe's Talk
- Two-loop integrand reduction method → Costas' Talk

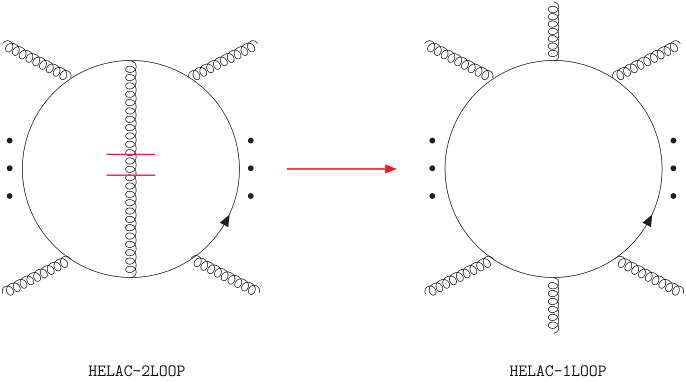
$$\mathcal{M}_n^{(L)} = \sum_F c_F^{(L)} \left(\sum_{i'} \tilde{c}_{i',F}^{(L)}\left(\{p_i \cdot p_j\},\{m\},\varepsilon\right) F_{i',F}^{(L)}\left(\{p_i \cdot p_j\},\{m\},\varepsilon\right) \right)$$

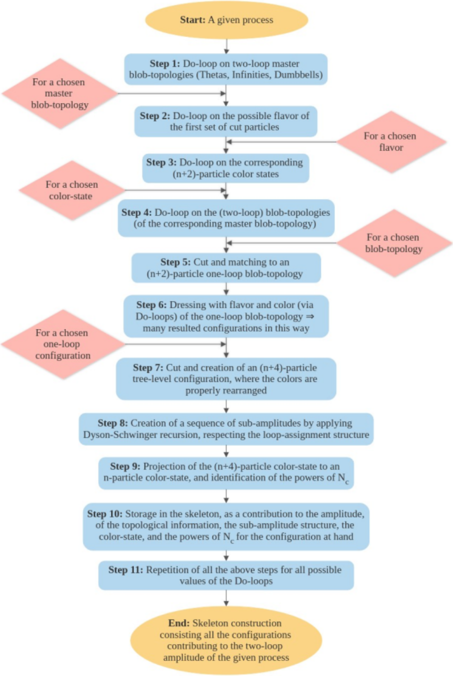
- Numerical implementation of the above method in `HELAC-2LOOP` → Aris' Talk

The Algorithm

Two-Loop Amplitude Construction using HELAC-2LOOP: The algorithm

n − *Particle & 2* − *Loop Amplitude* $\longrightarrow$ $(n+2)$ − *Particle & 1* − *Loop Amplitude*



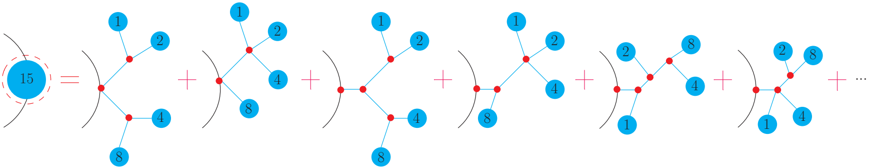

Start: A given process
Step 1: Do-loop on two-loop master blob-topologies (Thetas, Infinities, Dumbbells)
For a chosen master blob-topology
Step 2: Do-loop on the possible flavor of the first set of cut particles
For a chosen flavor
Step 3: Do-loop on the corresponding (n+2)-particle color states
For a chosen color-state
Step 4: Do-loop on the (two-loop) blob-topologies (of the corresponding master blob-topology)
For a chosen blob-topology
Step 5: Cut and matching to an (n+2)-particle one-loop blob-topology
Step 6: Dressing with flavor and color (via Do-loops) of the one-loop blob-topology ⇒ many resulted configurations in this way
For a chosen one-loop configuration
Step 7: Cut and creation of an (n+4)-particle tree-level configuration, where the colors are properly rearranged
Step 8: Creation of a sequence of sub-amplitudes by applying Dyson-Schwinger recursion, respecting the loop-assignment structure
Step 9: Projection of the (n+4)-particle color-state to an n-particle color-state, and identification of the powers of N_c
Step 10: Storage in the skeleton, as a contribution to the amplitude, of the topological information, the sub-amplitude structure, the color-state, and the powers of N_c for the configuration at hand
Step 11: Repetition of all the above steps for all possible values of the Do-loops
End: Skeleton construction consisting all the configurations contributing to the two-loop amplitude of the given process

Binary Representation and Blobs

- For the external particles we use a binary representation (2^{i-1}). E.g.:

$$\text{For } n = 4: \quad \{p_1, p_2, p_3, p_4\} \rightarrow \{1, 2, 4, 8\}.$$

- What a blob and its level are?



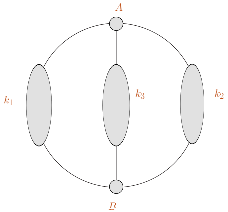
1) Blob $\rightarrow$ sum of all possible tree-level sub-currents that can be constructed.

2) Blob-Level $\rightarrow$ the number of particles consisted in the blob.

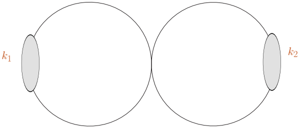
- In the example above: blob $\rightarrow$ 15, level $\rightarrow$ 4, and total number of graphs that describes $\rightarrow$ 26:
  - 3 graphs of the first type,
  - 4 graphs of the second type,
  - 3 graphs of the third type,
  - 4 graphs of the fourth type,
  - 12 graphs of the fifth type,
  - 3 graphs of the sixth type.

Two-loop blob-topologies

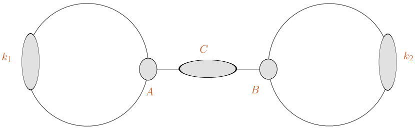
- At two-loop 3 master blob-topologies exist

Thetas: 

$\equiv \{\{k_1\},\{k_2\},\{k_3\},\{A\},\{B\}\}$

Infinities: 

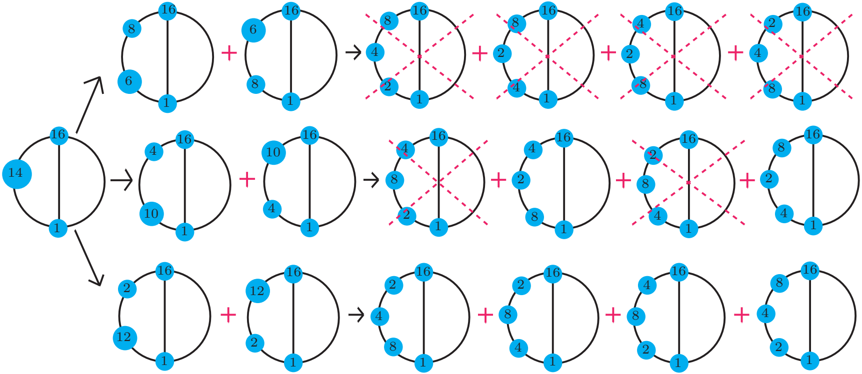
$\equiv \{\{k_1\},\{k_2\}\}$

Dumbbells: 

$\equiv \{\{k_1\},\{k_2\},\{C\},\{A\},\{B\}\}$

where with $\{x\}$ we denote a sub-list of blobs.

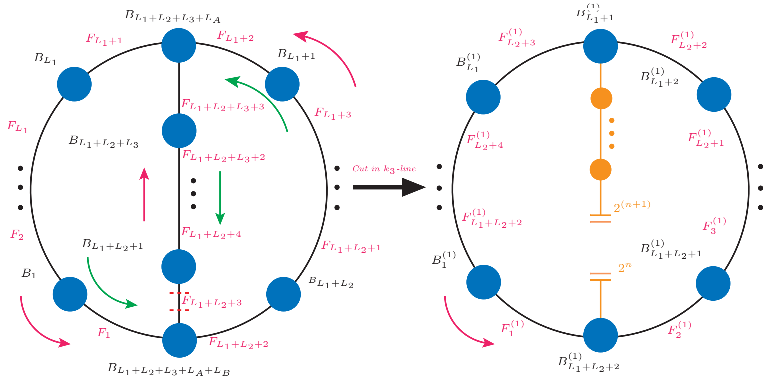
Blob-Topology Generation



- Generate of all possible blob-topologies: From higher to lower level blobs.
- Order in the Lengths: $L_1 \geq L_2 \geq L_3$ and $L_A \geq L_B$.
- Remove identical topologies using symmetries: 1) up-down (reversion), 2) loop-line swapping.

From 2-Loop to 1-Loop: Thetas

- The red (green) arrows in the following graphs indicate the flow of flavor (momenta):

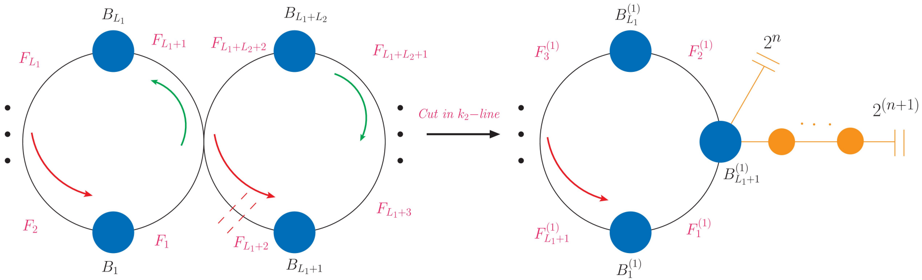


- The 1(2)-loop blobs (flavors) are defined algorithmically from the corresponding 2(1)-loop ones.
- Blob & flavor structure of the k_3-line and A/B-vertices stored → dictated recursion applied!

$$B^{(1)}_{L_1+L_2+2} = 2^n + B_{L_1+L_2+L_3+L_A+L_B} \quad \text{and} \quad B^{(1)}_{L_1+1} = 2^{n+1} + B_{L_1+L_2+1} + \cdots + B_{L_1+L_2+L_3} + B_{L_1+L_2+L_3+L_A}$$

From Two-Loop to One-Loop: The Infinity-Topologies

- The red (green) arrows in the following graphs indicate the flow of flavor (momenta):

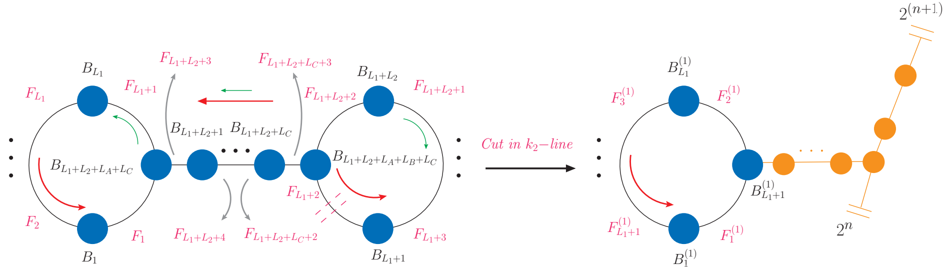


- The 1(2)-loop blobs (flavors) are defined algorithmically from the corresponding 2(1)-loop ones.
- Blob & flavor structure of k_2-line stored → dictated recursion applied!

$$B^{(1)}_{L_1+1} = 2^n + 2^{n+1} + B_{L_1+1} + \cdots + B_{L_1+L_2}$$

From Two-Loop to One-Loop: The Dumbbell-Topologies

- The red (green) arrows in the following graphs indicate the flow of flavor (momenta):
- The grey arrows indicate to which propagator the pointing flavor corresponds.



- The 1(2)-loop blobs (flavors) are defined algorithmically from the corresponding 2(1)-loop ones.
- Blob & flavor structure of k_2/C-lines and A/B-vertices stored → dictated recursion applied!

$$B^{(1)}_{L_1+1} = 2^n + 2^{n+1} + B_{L_1+1} + \cdots + B_{L_1+L_2+L_A+L_B+L_C}$$

Color-Flavor Dressing

- Current version: QCD particles on the loop + QCD & EW particles on the blobs!

- The color-flavor dressing of the blob-topologies occurs at 1-loop level with the following algorithm

Flavor Dressing:

- Identify the blob-flavors
- Assign flavor to the 1st propagator
- Apply Feynman rules in vertices

Color Dressing:

- Color of one-loop blobs already generated
- Assign color indices to the 1st propagator
- Track the color flow at each vertex

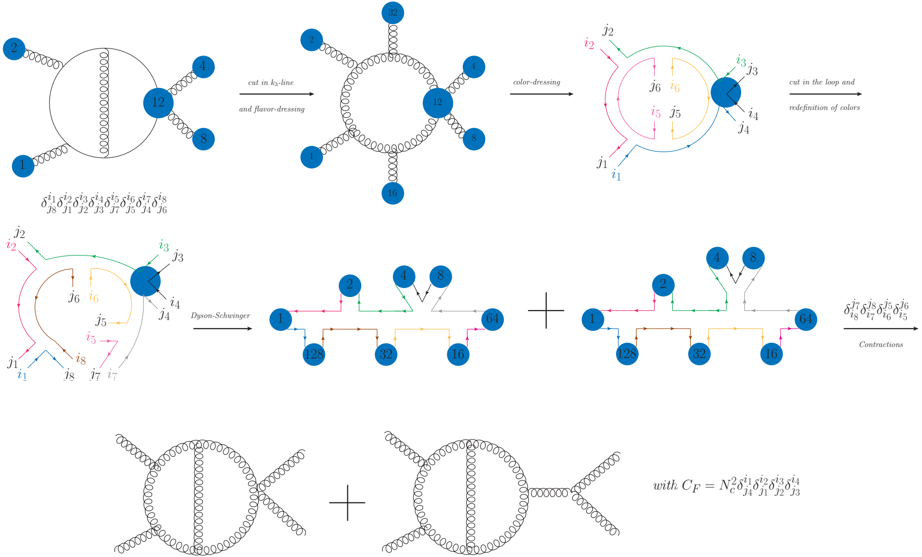
- The cut-lines are also dressed and stored for use when the Dyson-Schwinger recursion is applied.

- After the second cut, the flavor and the color of the new extra particles is defined and the color of the $n + 4$ particle configuration is rearranged by tracking the flow in the one-loop topology.

This way we obtain a unique configuration with specific color and flavor for each loop particle!

All contributing configurations obtained by iterating through all possible values of flavors and colors of the 1st propagator!

Construction: gluonic $\{\{1,2\},\{12\},\{\},\{\},\{\}\}$ with $\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}\delta^{i_5}_{j_6}\delta^{i_6}_{j_5}$



with $C_F = N_c^2\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}$

Skeleton: gluonic $\{\{1,2\},\{12\},\{\},\{\},\{\}\}$ *with* $\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}$

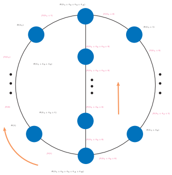
Output in the usual HELAC notation, except for the last line!

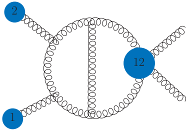
```
INFO ==========================================
INFO COLOR           9 out of          24
INFO number of nums          0
INFO ==========================================
INFO COLOR          10 out of          24
INFO number of nums        208
 INFO NUM           52  of           208           7
 INFO ====================================
INFO   4  80  35   9   1   1  16  35   5  64  35   7   0   0   0   0   1   2
INFO   4  12  35  10   1   1   4  35   3   8  35   4   0   0   0   0   1   1
INFO   4  92  35  11   1   2  12  35  10  80  35   9   0   0   0   0   1   1
INFO   5  92  35  11   2   2   4  35   3   8  35   4  80  35   9   0   1   5
INFO   4 124  35  12   1   1  32  35   6  92  35  11   0   0   0   0   1   2
INFO   4 126  35  13   1   1   2  35   2 124  35  12   0   0   0   0   1   1
INFO   4 254  35  14   1   1 128  35   8 126  35  13   0   0   0   0   1   2
INFO   6   1  12   1   2  12  35  35  35  35  35  35   0   0   0   0  99   9
```

→ Skeleton stored color-wise

$$\mathrm{ID}_{topo} = \begin{cases} (2)^{L_1}(3)^{L_2}(5)^{L_3}(7)^{L_A}(11)^{L_B}, & \text{Theta} \\ (2)^{L_1}(3)^{L_2}, & \text{Infinity} \\ (2)^{L_1}(3)^{L_2}(5)^{L_C}(7)^{L_A}(11)^{L_B}, & \text{Dumbbell} \end{cases}$$

$$mb = \begin{cases} 1, & \text{Theta} \\ 2, & \text{Infinity} \\ 3, & \text{Dumbbell} \end{cases}$$





Skeleton knows nothing about d (it can be used in $d = 4$ or $d = 4 - 2\varepsilon$)!

Results

Skeleton Results (Thetas) for Several Processes

Process	Loop-Flavors	Time	Numerators
$pp \to ggg$	$\{g, q, \bar{q}, c, \bar{c}\}$	34368 s	1853960
$pp \to t\bar{t}j$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	15559 s	306240
$pp \to ggH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	3849 s	67696
$pp \to t\bar{t}H$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1962 s	18708
$pp \to HHg$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1350 s	22676
$pp \to HHH$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	259 s	9096
$pp \to ggZ/\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	5302 s	176636
$pp \to t\bar{t}Z/\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	2035 s	66116
$pp \to gZZ/\gamma\gamma/Z\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1809 s	58744
$pp \to ZZZ/\gamma\gamma\gamma/Z\gamma\gamma/ZZ\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	491 s	23096
$pp \to gHZ/\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	1484 s	23524
$pp \to HZZ/\gamma\gamma/Z\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	293 s	9192
$pp \to HHZ/\gamma$	$\{g, q, \bar{q}, t, \bar{t}, c, \bar{c}\}$	255 s	8912
$u\bar{d} \to ggW^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	1592 s	45508
$u\bar{d} \to t\bar{t}W^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	142 s	5692
$pp \to gW^+W^-$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	1723 s	25720
$u\bar{d} \to gHW^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	318 s	8468
$u\bar{d} \to HHW^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	37 s	200
$u\bar{d} \to W^+W^-W^+$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	81 s	2140
$u\bar{d} \to gW^+Z/\gamma$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	239 s	9860
$u\bar{d} \to HW^+Z/\gamma$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	44 s	264
$u\bar{d} \to W^+ZZ/\gamma\gamma/Z\gamma$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	70 s	3040
$pp \to W^+W^-Z/\gamma$	$\{g, u, \bar{u}, d, \bar{d}, t, \bar{t}, c, \bar{c}\}$	222 s	6464

Outcome

• Presented an algorithmic construction of two-loop integrand numerators using a hybrid Dyson-Schwinger recursion within the `HELAC` framework!

• Showed several results for $2 \to 3$ processes consisting of EW and QCD external particles.

• Cross-checked some of the numerators if these processes against FeynArts [Hahn, 2000] and FeynCalc [Shtabovenko, Mertig & Orellana, 2023].

What's Next

• Optimization of the code and the skeleton information (combine identical numerators) for optimizing the efficiency of the evaluations.

• Incorporation of the d-dimensional evaluation of the skeleton (see Giuseppe's talk) & the integrand reduction technique of [Bevilacqua, Canko, Papadopoulos, Spourdalakis, 2025] (see Costas' and Aris' talks).

Thank you very much!

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H.F.R.I.
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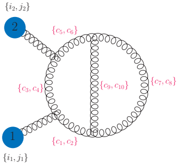
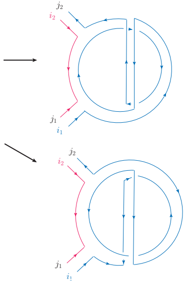
Backup-Slides: Identical Color Configurations for **HELAC**



$\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$



$\{c_1, c_2\} = \{i_1, i_1\}$
$\{c_3, c_4\} = \{i_2, i_1\}$
$\{c_5, c_6\} = \{i_1, i_1\}$
$\{c_7, c_8\} = \{i_1, i_1\}$
$\{c_9, c_{10}\} = \{i_1, i_1\}$

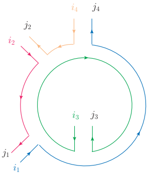
$C_F = -\delta^{i_1}_{c_1}\delta^{c_1}_{c_7}\delta^{c_7}_{c_{10}}\delta^{c_{10}}_{c_2}\delta^{c_2}_{c_4}\delta^{c_4}_{c_6}\delta^{c_6}_{c_8}\delta^{c_8}_{c_9}\delta^{c_9}_{c_5}\delta^{c_5}_{j_2}\delta^{i_2}_{c_3}\delta^{c_3}_{j_1}$

$\{c_1, c_2\} = \{i_1, i_1\}$
$\{c_3, c_4\} = \{i_2, i_1\}$
$\{c_5, c_6\} = \{i_1, i_1\}$
$\{c_7, c_8\} = \{i_1, i_1\}$
$\{c_9, c_{10}\} = \{i_1, i_1\}$

$C_F = -\delta^{i_1}_{c_1}\delta^{c_1}_{c_9}\delta^{c_9}_{c_8}\delta^{c_8}_{c_2}\delta^{c_2}_{c_4}\delta^{c_4}_{c_6}\delta^{c_6}_{c_{10}}\delta^{c_{10}}_{c_7}\delta^{c_7}_{c_5}\delta^{c_5}_{j_2}\delta^{i_2}_{c_3}\delta^{c_3}_{j_1}$

This is not any more the case after cutting in k_3-line:

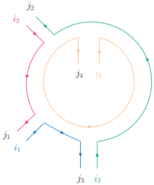
1)



$\{c_1, c_2\} = \{i_1, i_3\}$
$\{c_3, c_4\} = \{i_2, i_3\}$
$\{c_5, c_6\} = \{i_4, i_3\}$
$\{c_7, c_8\} = \{i_1, i_3\}$

$\delta^{i_1}_{j_4}\delta^{i_2}_{j_1}\delta^{i_3}_{j_3}\delta^{i_4}_{j_2}$

2)



$\{c_1, c_2\} = \{i_1, i_4\}$
$\{c_3, c_4\} = \{i_2, i_4\}$
$\{c_5, c_6\} = \{i_3, i_4\}$
$\{c_7, c_8\} = \{i_3, i_4\}$

$\delta^{i_1}_{j_3}\delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_4}$

- Theta-Topology symmetries[1]:

$$\begin{aligned}\{\{k_1\},\{k_2\},\{k_3\},\{A\},\{B\}\} &= \{R[\{k_1\}],R[\{k_2\}],R[\{k_3\}],\{B\},\{A\}\} = \{\{k_1\},\{k_3\},\{k_2\},\{A\},\{B\}\}\\ &= \{R[\{k_1\}],R[\{k_3\}],R[\{k_2\}],\{B\},\{A\}\} = \{\{k_2\},\{k_1\},\{k_3\},\{A\},\{B\}\}\\ &= \{R[\{k_2\}],R[\{k_1\}],R[\{k_3\}],\{B\},\{A\}\} = \{\{k_2\},\{k_3\},\{k_1\},\{A\},\{B\}\}\\ &= \{R[\{k_2\}],R[\{k_3\}],R[\{k_1\}],\{B\},\{A\}\} = \{\{k_3\},\{k_1\},\{k_2\},\{A\},\{B\}\}\\ &= \{R[\{k_3\}],R[\{k_1\}],R[\{k_2\}],\{B\},\{A\}\} = \{\{k_3\},\{k_2\},\{k_1\},\{A\},\{B\}\}\\ &= \{R[\{k_3\}],R[\{k_2\}],R[\{k_1\}],\{B\},\{A\}\}\end{aligned}$$

- Infinity-Topology symmetries:

$$\begin{aligned}\{\{k_1\},\{k_2\}\} &= \{\{k_2\},\{k_1\}\} = \{R[\{k_1\}],\{k_2\}\} = \{\{k_1\},R[\{k_2\}]\} = \{R[\{k_1\}],R[\{k_2\}]\}\\ &= \{R[\{k_2\}],\{k_1\}\} = \{\{k_2\},R[\{k_1\}]\} = \{R[\{k_2\}],R[\{k_1\}]\}\end{aligned}$$

- Dumbbell-Topology symmetries:

$$\begin{aligned}\{\{k_1\},\{k_2\},\{C\},\{A\},\{B\}\} &= \{R[\{k_1\}],\{k_2\},\{C\},\{A\},\{B\}\} = \{\{k_1\},R[\{k_2\}],\{C\},\{A\},\{B\}\}\\ &= \{R[\{k_1\}],R[\{k_2\}],\{C\},\{A\},\{B\}\} = \{\{k_2\},\{k_1\},R[\{C\}],\{B\},\{A\}\}\\ &= \{R[\{k_2\}],\{k_1\},R[\{C\}],\{B\},\{A\}\} = \{\{k_2\},R[\{k_1\}],R[\{C\}],\{B\},\{A\}\}\\ &= \{R[\{k_2\}],R[\{k_1\}],R[\{C\}],\{B\},\{A\}\}\end{aligned}$$

[1] We use the notation $R[\{k_i\}] := Reverse[\{k_i\}]$.