NATIONAL CENTRE FOR
SCIENTIFIC RESEARCH "DEMOKRITOS"
INSTITUTE OF NUCLEAR AND PARTICLE PHYSICS

H.F.R.I.
Hellenic Foundation for
Research & Innovation

Towards numerical computation of dimensionally regularised QCD helicity amplitudes

Giuseppe Bevilacqua
NCSR "Demokritos"

INPP Annual Meeting 2024

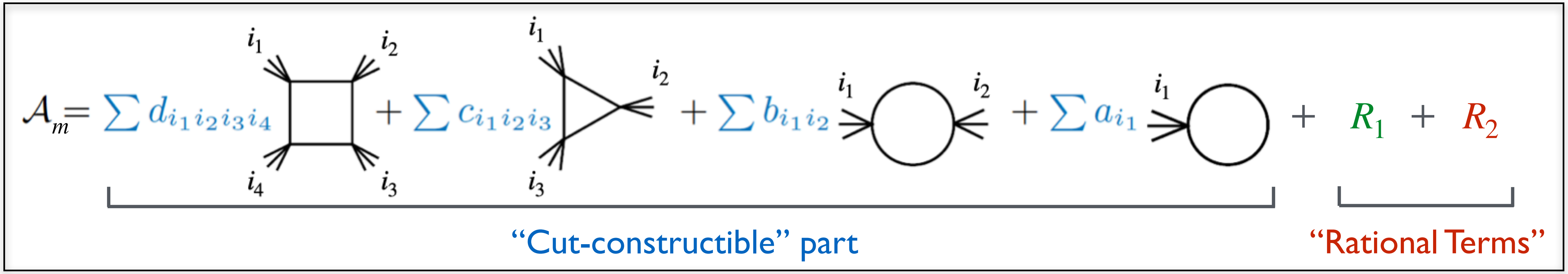
March 28, 2024

In collaboration with D. Canko, N. Dokmetzoglou, C. Papadopoulos and A. Spourdalakis

Motivation

$$\mathscr{A}_m = \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q})}{\bar{D}_0\,\bar{D}_1 \ldots \bar{D}_{m-1}}$$

One-loop reduction in a nutshell



“Cut-constructible” part

“Rational Terms”

- Key input of OPP reduction is the numerator of loop integrand
  - Typically, ME generators provide numerators in $d = 4$ → $N(q)$
  - Mismatch with the d-dimensional quantity appearing in loop integrand → $\bar{N}(\bar{q})$
  - Rational Terms (R_1, R_2) compensate for the mismatch in D_i’s and $N(q)$
- Achieving numerical computation of d-dimensional amplitudes has potential in performance and bookkeeping (→ no need for R_1, R_2)
- Beyond 1-loop: more natural treatment of loop reduction → Aris’ talk
- Starting point: feasibility study at 1-loop in the HELAC framework

Outline

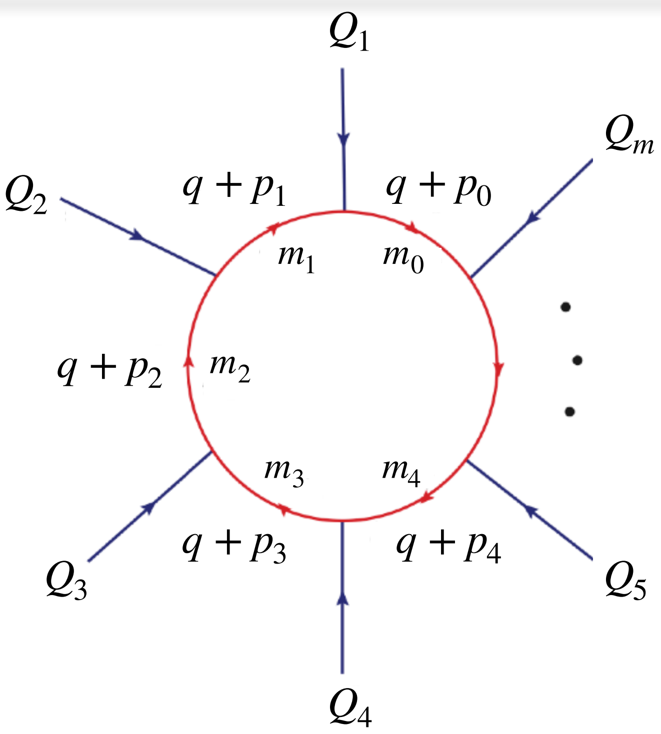
- Warm-up example: *analytical* structure in d dimensions
- Organisation of numerical calculation: Dyson-Schwinger recursion
- Obtaining full d-dimensional results: proof of concept

Basic notation

$$\mathcal{A}_m = \int \frac{d^d\bar{q}}{(2\pi)^d} \frac{\bar{N}(\bar{q})}{\bar{D}_0\,\bar{D}_1 \ldots \bar{D}_{m-1}}$$

$\bar{N}(\bar{q}) \longrightarrow$ *Numerator*

$\bar{D}_i \equiv (\bar{q} + p_i)^2 - m_i^2 \longrightarrow$ *Denominators*



$$\bar{q}^2 = q^2 + \underset{\displaystyle\mu}{\underset{\equiv}{\tilde{q}^2}}$$

$$\bar{\gamma}^\mu = \gamma^\mu + \tilde{\gamma}^\mu$$

$$\hookrightarrow \; \bar{\gamma}^\mu\,\bar{\gamma}_\mu = d = 4 - 2\epsilon$$

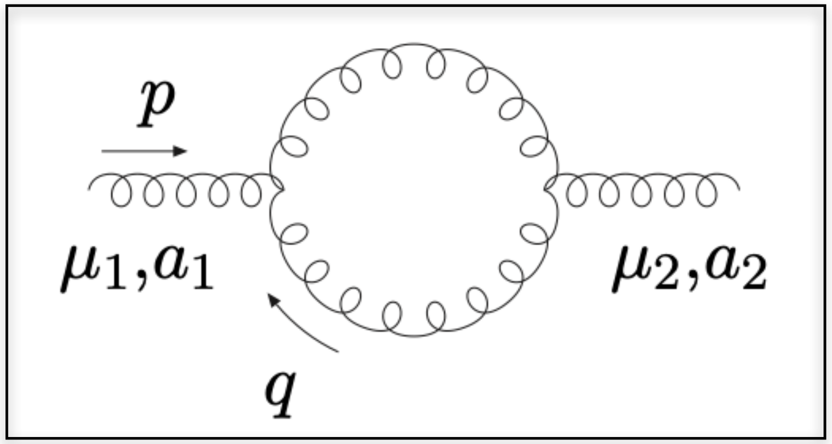
$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}$$

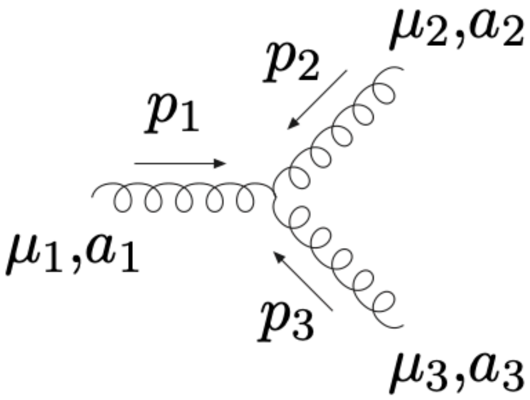
$$\hookrightarrow \; \bar{g}^{\mu\nu}\,\bar{g}_{\mu\nu} = d = 4 - 2\epsilon$$

$$\bar{D}_i^2 = D_i^2 + \mu \qquad \bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q})$$

- Dimensional regularisation → 't Hooft-Veltman scheme
  - physical momenta $(Q_1, Q_2, \ldots, Q_m)$ in $d = 4$
  - loop momentum $(\bar{q})$ in $d = 4 - 2\epsilon$

Warm-up example: 1-loop gluon self-energy





$$= g\, f^{a_1a_2a_3}\, \underbrace{V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)}_{\begin{array}{c}\rotatebox{90}{$\equiv$}\\ g_{\mu_1\mu_2}(p_2-p_1)_{\mu_3} \\ +\,g_{\mu_2\mu_3}(p_3-p_2)_{\mu_1} \\ +\,g_{\mu_3\mu_1}(p_1-p_3)_{\mu_2}\end{array}}$$

$$\bar N(\bar q) \;=\; \bar N^{\mu_1\mu_2}(\bar q)\;\varepsilon_{\mu_1}(p)\;\varepsilon_{\mu_2}(p) \;=\; N(q) \;+\; \tilde N(\bar q)$$

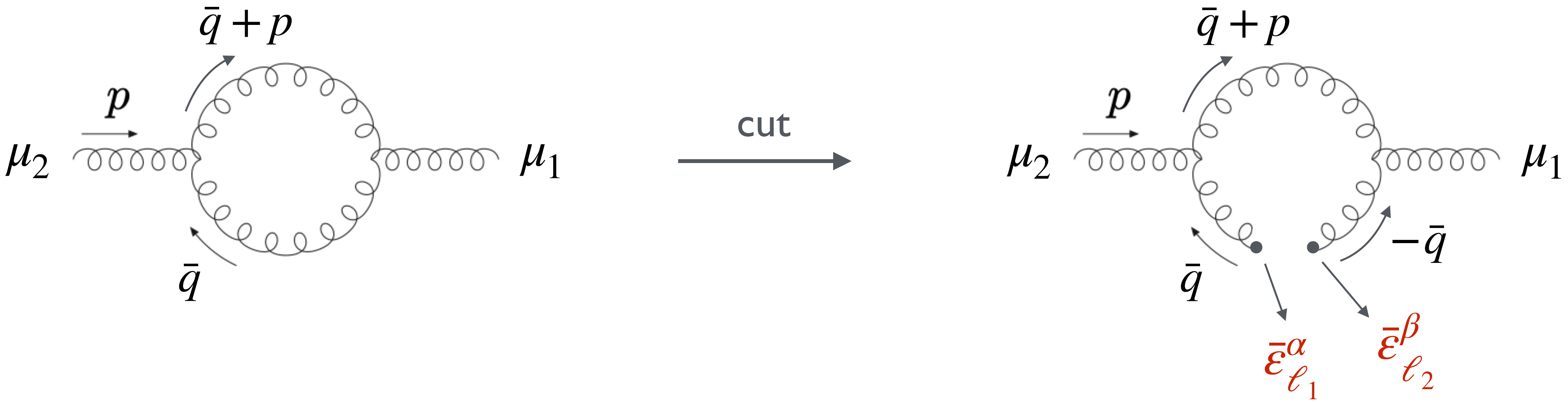
$$\hookrightarrow \quad \bar N^{\mu_1\mu_2}(\bar q) \;\to\; V^{\mu_1}{}_{\bar\beta\bar\gamma}(p,\,-\bar q-p,\,\bar q)\; V^{\mu_2\,\bar\gamma\bar\beta}(-p,\,-\bar q,\,\bar q+p) \;=$$

$$\boxed{\begin{aligned} = \;& \left(5p^2+2p\cdot q+2q^2\right) g^{\mu_1\mu_2} - 2\,p^{\mu_1}p^{\mu_2} + 5\,p^{\mu_1}q^{\mu_2} + 5\,q^{\mu_1}p^{\mu_2} + 10\,q^{\mu_1}q^{\mu_2} \\ & -\epsilon\left(2p^{\mu_1}p^{\mu_2} + 4p^{\mu_1}q^{\mu_2} + 4q^{\mu_1}p^{\mu_2} + 8q^{\mu_1}q^{\mu_2}\right) + 2\,\mu\, g^{\mu_1\mu_2} \end{aligned}} \quad \begin{aligned} &\longrightarrow \; N(q) \\ &\longrightarrow \; \tilde N(\bar q) \end{aligned}$$

Extra-dimensional terms $\propto \mu, \epsilon$

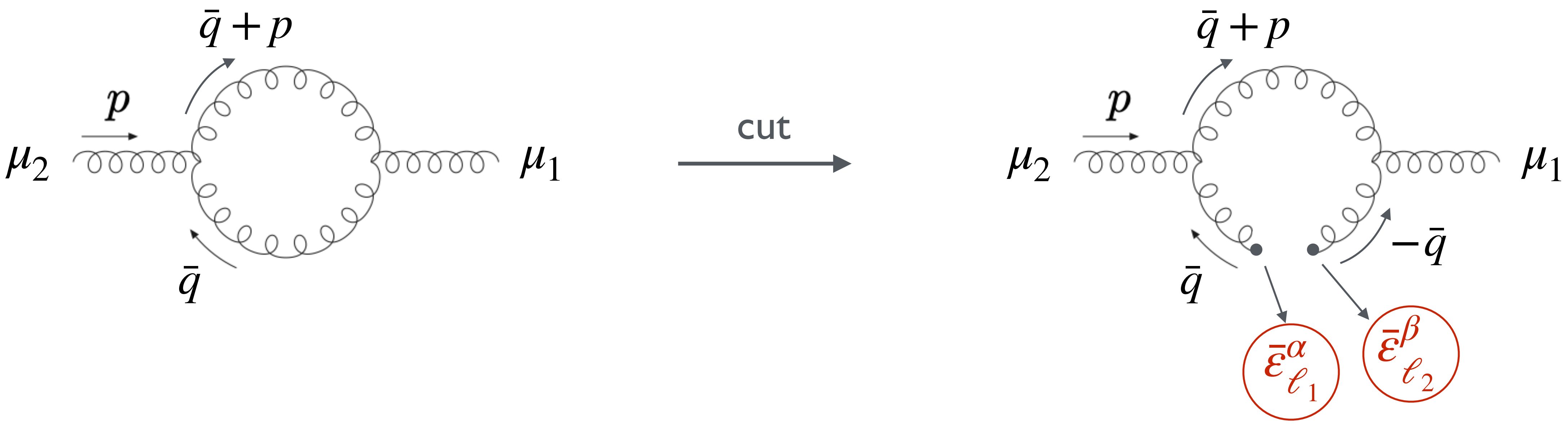
Organisation of numerical calculations

- Cutting loop propagator → Tree-level process with two extra particles



Organisation of numerical calculations

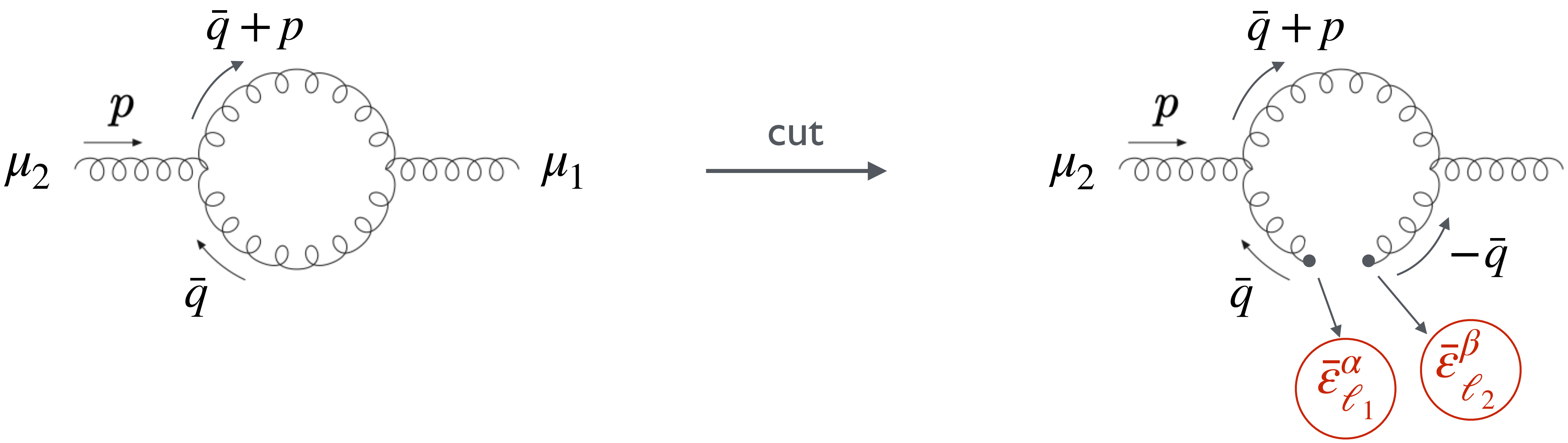
- Cutting loop propagator → Tree-level process with two extra particles



$$\hookrightarrow \boxed{\sum_{\lambda} \bar\varepsilon^{\alpha}_{\ell_1,\lambda}\,\bar\varepsilon^{\beta}_{\ell_2,\lambda} \to \bar g^{\alpha\beta}}$$

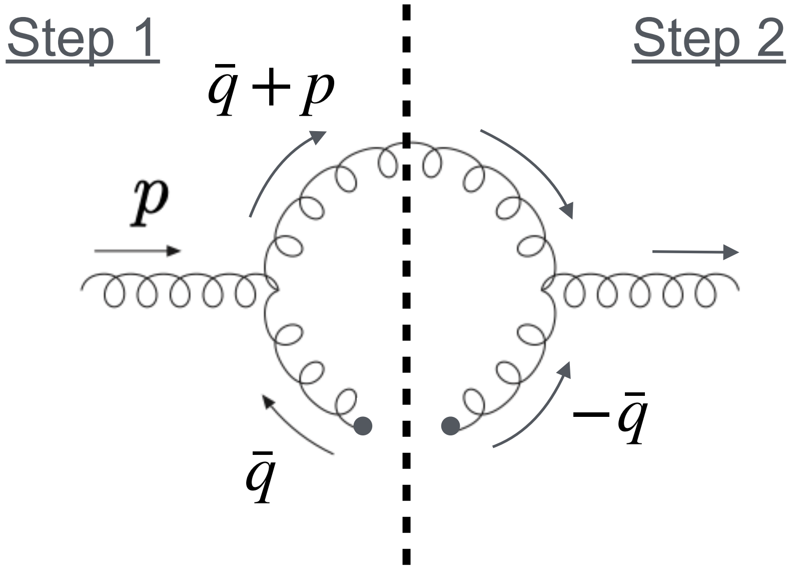
Organisation of numerical calculations

- Cutting loop propagator → Tree-level process with two extra particles



↪ $$\boxed{\sum_{\lambda} \bar{\varepsilon}^{\alpha}_{\ell_1,\lambda}\,\bar{\varepsilon}^{\beta}_{\ell_2,\lambda} \to \bar{g}^{\alpha\beta}}$$

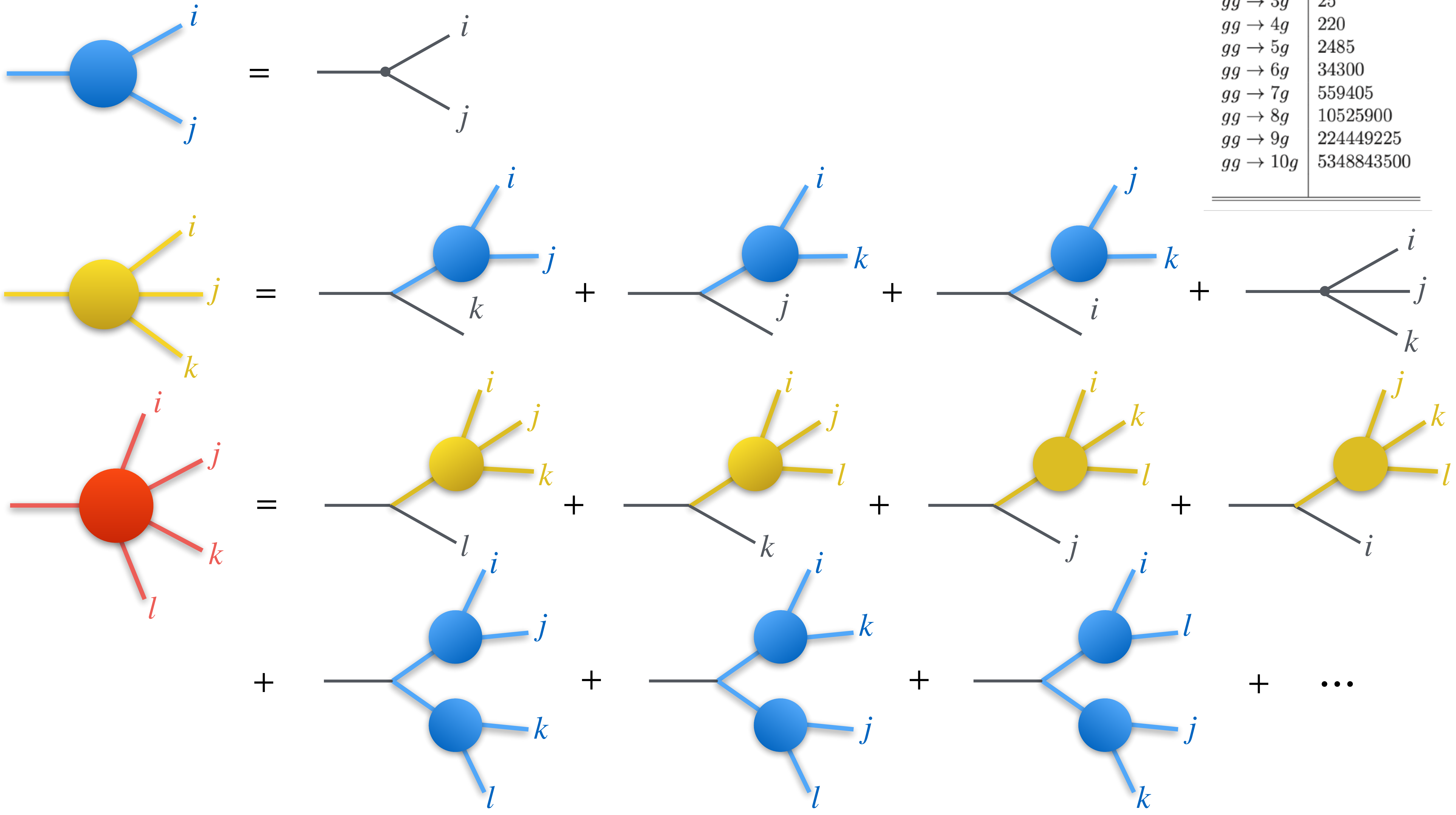
- Recursive calculation of tree-level process



Dyson-Schwinger recursion in a nutshell

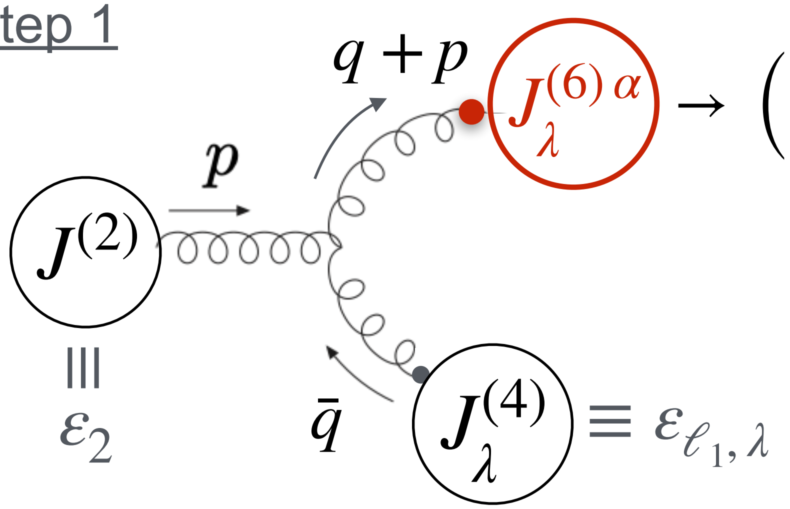
- Computing scattering amplitudes without Feynman diagrams

Process	N. diagrams
$gg \to 2g$	4
$gg \to 3g$	25
$gg \to 4g$	220
$gg \to 5g$	2485
$gg \to 6g$	34300
$gg \to 7g$	559405
$gg \to 8g$	10525900
$gg \to 9g$	224449225
$gg \to 10g$	5348843500



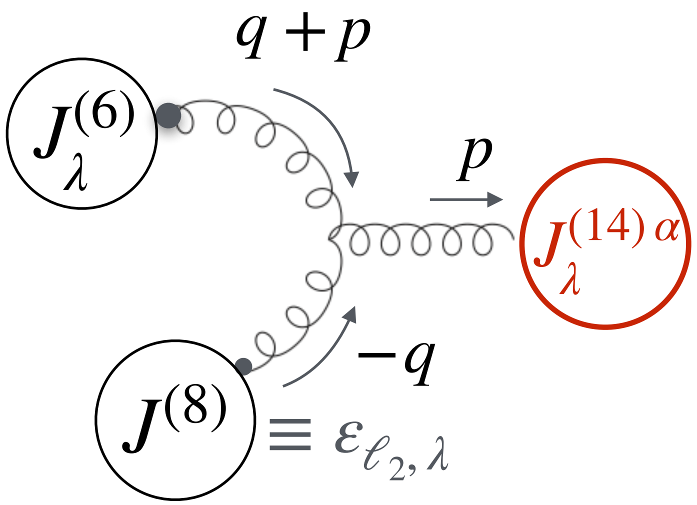
Recursive calculation in $d = 4$

Step 1



$$\to \Big((-q - 2\,p)\cdot \varepsilon_{\ell_1,\lambda}\Big)\, J^{(2)\,\alpha} + \big(-p\cdot J_2\big)\, \varepsilon^{\alpha}_{\ell_1,\lambda} + \Big(J_2 \cdot \varepsilon_{\ell_1,\lambda}\Big)\,(p - q)^{\alpha}$$

Step 2



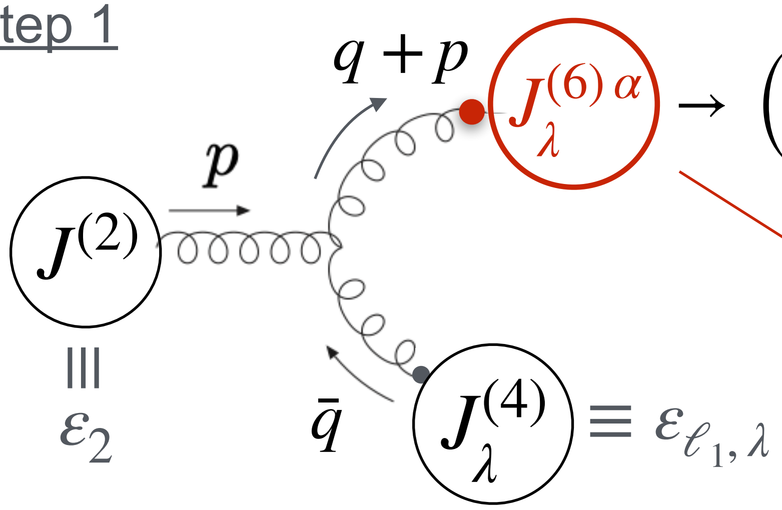
$$\to \Big((-q - 2\,p)\cdot \varepsilon_{\ell_2,\lambda}\Big)\, J^{(6)\,\alpha} + \big((-q - p)\cdot J^{(6)}\big)\, \varepsilon^{\alpha}_{\ell_2,\lambda} + \Big(J^{(6)} \cdot \varepsilon_{\ell_2,\lambda}\Big)\,(p + 2\,q)^{\alpha}$$

↪ $$\boxed{N(q) = \sum_{\lambda} \Big(J^{(14)}_{\lambda}\cdot \epsilon_1\Big)}$$

Full numerator in $d = 4$ dimensions

Recursive calculation in $d = 4$

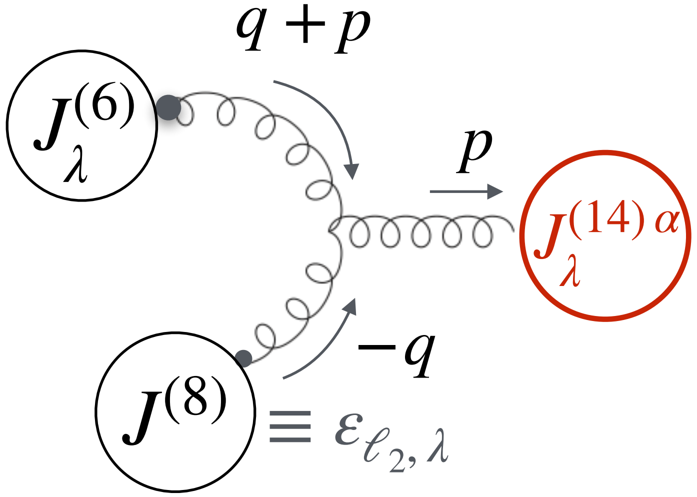
Step 1



$$\to \left((-q - 2\,p)\cdot\varepsilon_{\ell_1,\lambda}\right) J^{(2)\,\alpha} + \left(-p\cdot J_2\right)\varepsilon^{\alpha}_{\ell_1,\lambda} + \left(J_2\cdot\varepsilon_{\ell_1,\lambda}\right)(p - q)^{\alpha}$$

$$J^{(n_1+n_2)\,\alpha} = V_3^{\alpha}(\,J^{(n_1)}\,,J^{(n_2)}\,)$$

Step 2



$$\to \left((-q - 2\,p)\cdot\varepsilon_{\ell_2,\lambda}\right) J^{(6)\,\alpha} + \left((-q - p)\cdot J^{(6)}\right)\varepsilon^{\alpha}_{\ell_2,\lambda} + \left(J^{(6)}\cdot\varepsilon_{\ell_2,\lambda}\right)(p + 2\,q)^{\alpha}$$

$$\hookrightarrow \quad \boxed{N(q) = \sum_{\lambda}\left(J^{(14)}_{\lambda}\cdot\epsilon_1\right)}$$

Full numerator in $d = 4$ dimensions

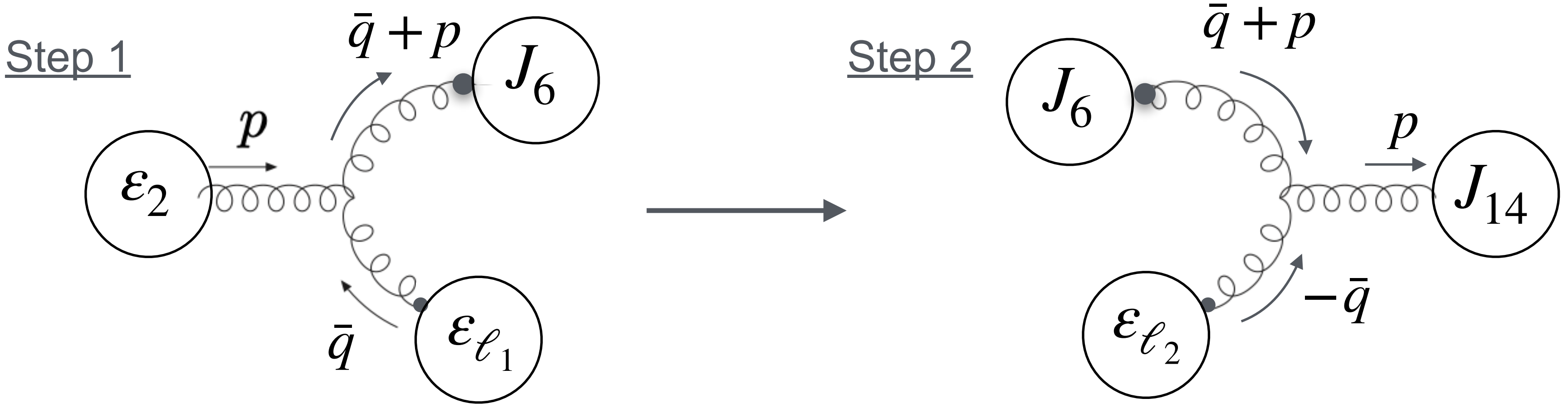
Moving to d dimensions

- Obtaining the full d-dimensional expression, $\bar{N}(\bar{q})$, from $N(q)$ is straightforward *analytically*. Just use the following rules in the expressions derived before:

$$q^2 X \to (q^2 + \mu) X \qquad \sum_{\lambda} (\varepsilon_{\ell_1,\lambda} \cdot \varepsilon_{\ell_2,\lambda})\, X \to \sum_{\lambda} (\varepsilon_{\ell_1,\lambda} \cdot \varepsilon_{\ell_2,\lambda})\, X + (d-4)\, X$$

- This approach is possible if one works with analytic expressions. It is not as simple when *numerical* methods are adopted.

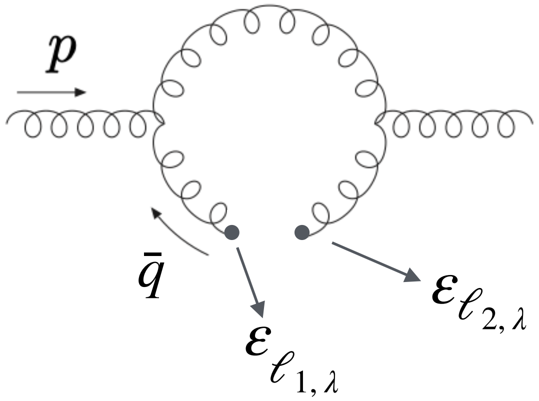
↪ Numerical recursion keeps "memory" of the previous step only, not of the full recursion history



Moving to d dimensions

- The missing terms required to match the full d-dimensional result, $\bar{N}(\bar{q})$, can be generated numerically if the following prescriptions can be applied throughout the recursion:

Structure in $d = 4$	Extra term
$q^2 X$ $\longrightarrow$	μX
$\sum_{\lambda} (\varepsilon_{\ell_1,\lambda} \cdot \varepsilon_{\ell_2,\lambda})\, X$ $\longrightarrow$	$(d-4)\, X$
$\sum_{\lambda} (q \cdot \varepsilon_{\ell_1,\lambda})\,(q \cdot \varepsilon_{\ell_2,\lambda})\, X$ $\longrightarrow$	$(q^2 + \mu)\, X$



$$\bar{q}^2 = q^2 + \mu$$

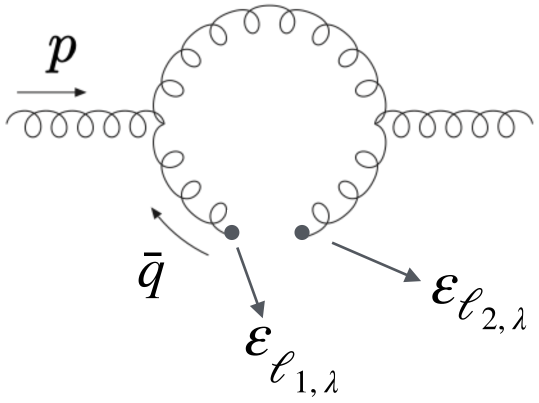
$$\sum_{\lambda} \bar{\varepsilon}^{\alpha}_{\ell_1,\lambda}\, \bar{\varepsilon}^{\beta}_{\ell_2,\lambda} \rightarrow \bar{g}^{\alpha\beta}$$

Moving to d dimensions

- The missing terms required to match the full d-dimensional result, $\bar{N}(\bar{q})$, can be generated numerically if the following prescriptions can be applied throughout the recursion:

Structure in $d = 4$		Extra term
$q^2 X$	$\longrightarrow$	μX
$\sum_{\lambda} (\varepsilon_{\ell_1,\lambda} \cdot \varepsilon_{\ell_2,\lambda}) X$	$\longrightarrow$	$(d-4) X$
$\sum_{\lambda} (q \cdot \varepsilon_{\ell_1,\lambda})(q \cdot \varepsilon_{\ell_2,\lambda}) X$	$\longrightarrow$	$(q^2 + \mu) X$

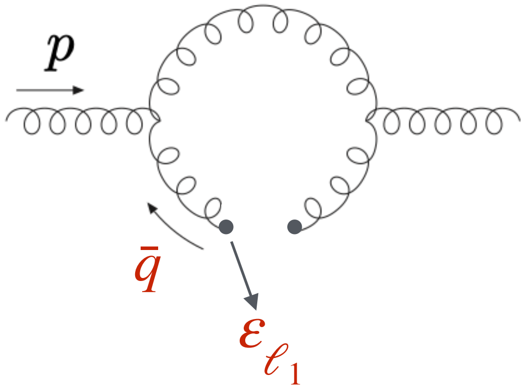
We need to keep track of these structures (in particular of their *coefficients*) inside currents (J^{α})



$$\bar{q}^2 = q^2 + \mu$$

$$\sum_{\lambda} \bar{\varepsilon}^{\alpha}_{\ell_1,\lambda} \bar{\varepsilon}^{\beta}_{\ell_2,\lambda} \rightarrow \bar{g}^{\alpha\beta}$$

Moving to d dimensions



$$J^\alpha \equiv \boxed{C_q}\, q^\alpha + \boxed{C_\varepsilon}\, \varepsilon^\alpha_{\ell_1} + \boxed{J^\alpha_{q\varepsilon}}\, (q \cdot \varepsilon_{\ell_1}) + R^\alpha$$

$$\downarrow$$

$$J^\alpha_{q\varepsilon} \equiv \boxed{C^{(q)}_{q\varepsilon}}\, q^\alpha + R'^\alpha$$

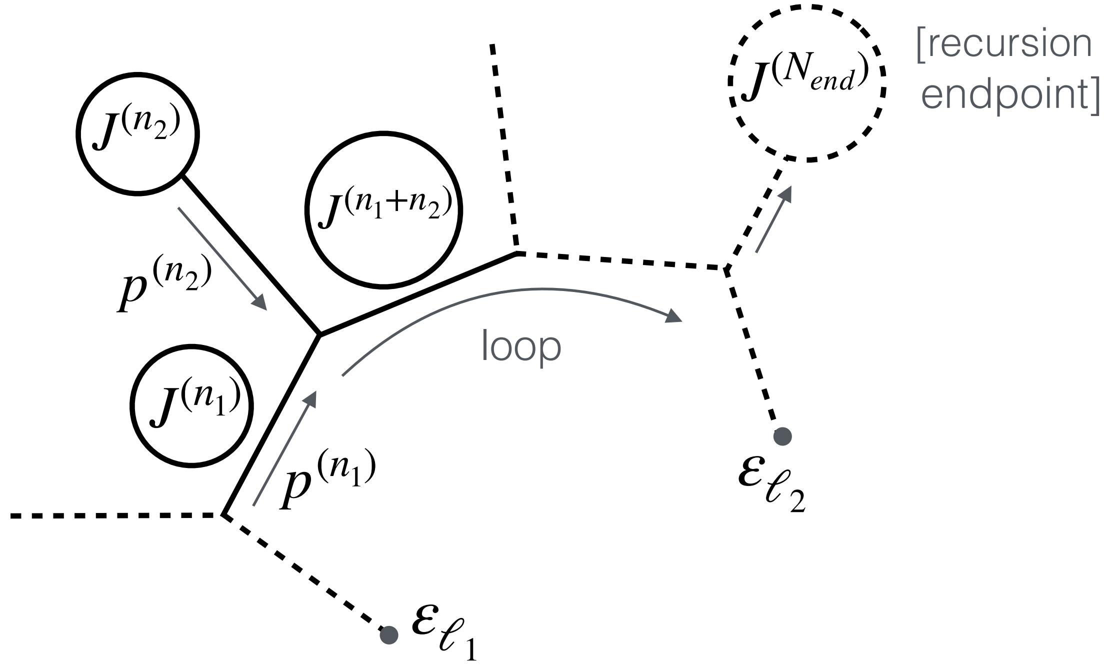
- Claims

  - Keeping track of coefficients C_q , C_ε , $C^{(q)}_{q\varepsilon}$ (scalars) and $J^\alpha_{q\varepsilon}$ (vector) at every step of the recursion *in addition to* J^α is sufficient to obtain full $d-$dimensional numerators for gluon loops

  - The coefficients above can be computed *recursively* (i.e., no need of analytic expressions to keep track of them)

We'll show the recursion relations for the coefficients and how to combine them to obtain the full d-dimensional numerator

Recursion relations



Initial conditions*

$C_q^{(1)} = 0 \qquad C_\varepsilon^{(1)} = 1$

$C_{q\varepsilon}^{(1)\,(q)} = 0 \qquad J_{q\varepsilon}^{(1)\ \alpha} = 0$

* valid for level 1 of recursion

$N_{end} \equiv$ recursion endpoint

$$J^{(n_1+n_2)\,\alpha} = V_3^\alpha\left(J^{(n_1)}, J^{(n_2)}\right) - \mu\left(1 - 2\,\delta_{(n_1+n_2)\,N_{end}}\right) C_q^{(n_1)}\, J^{(n_2)}$$

$$C_q^{(n_1+n_2)} = C_q^{(n_1)}\left(2\,p^{(n_1)} + p^{(n_2)}\right)\cdot J^{(n_2)} - \left(1 + \delta_{(n_1+n_2)\,N_{end}}\right) J^{(n_1)}\cdot J^{(n_2)}$$

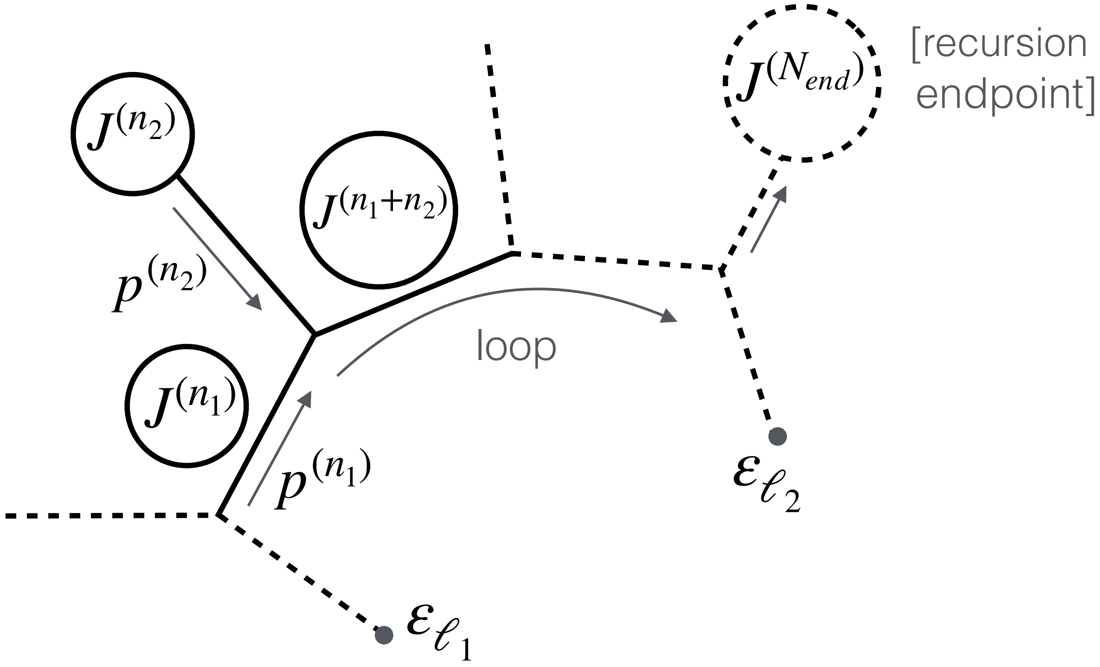
$$C_\varepsilon^{(n_1+n_2)} = C_\varepsilon^{(n_1)}\left(2\,p^{(n_1)} + p^{(n_2)}\right)\cdot J^{(n_2)}$$

$$C_{q\varepsilon}^{(n_1+n_2)\,(q)} = C_{q\varepsilon}^{(n_1)\,(q)}\left(2\,p^{(n_1)} + p^{(n_2)}\right)\cdot J^{(n_2)} - \left(1 + \delta_{(n_1+n_2)\,N_{end}}\right) J_{q\varepsilon}^{(n_1)}\cdot J^{(n_2)}$$

$$J_{q\varepsilon}^{(n_1+n_2)\ \alpha} = V_3^\alpha\left(J_{q\varepsilon}^{(n_1)}, J^{(n_2)}\right) - \left(C_\varepsilon^{(n_1+n_2)} + \mu\, C_{q\varepsilon}^{(n_1)\,(q)}\right) J^{(n_2)\,\alpha}$$

$$\hookrightarrow \quad J_{q\varepsilon}^{(N_{end})\ \alpha} = J_{q\varepsilon}^{(n_1)\ \alpha} + C_{q\varepsilon}^{(n_1)\,(q)}\left(p^{(n_2)} - p^{(n_1)}\right)^\alpha$$

Recursion relations for coefficients



Initial conditions*

$C_q^{(1)} = 0$ $\quad C_\varepsilon^{(1)} = 1$

$C_{q\varepsilon}^{(1)\,(q)} = 0$ $\quad J_{q\varepsilon}^{(1)\;\alpha} = 0$

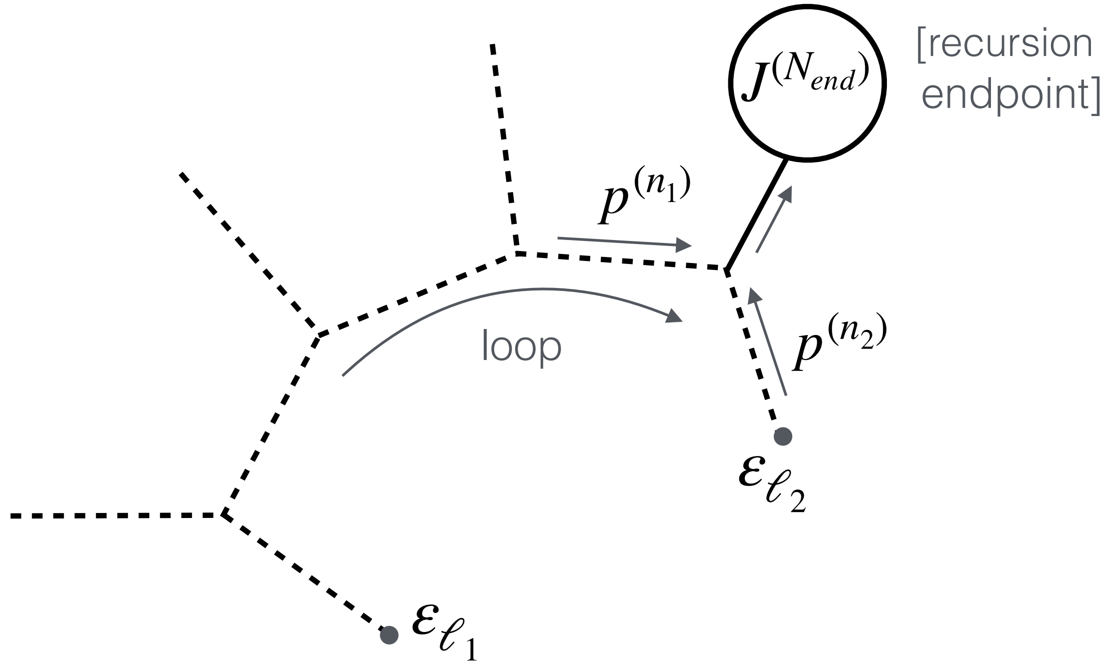
* valid for level 1 of recursion

$$J^{(n_1+n_2)\,\alpha} = V_3^\alpha\,(\,J^{(n_1)}, J^{(n_2)}\,)$$

4-dimensional limit:

$d \to 4, \quad \mu \to 0$

Full numerator in d dimensions

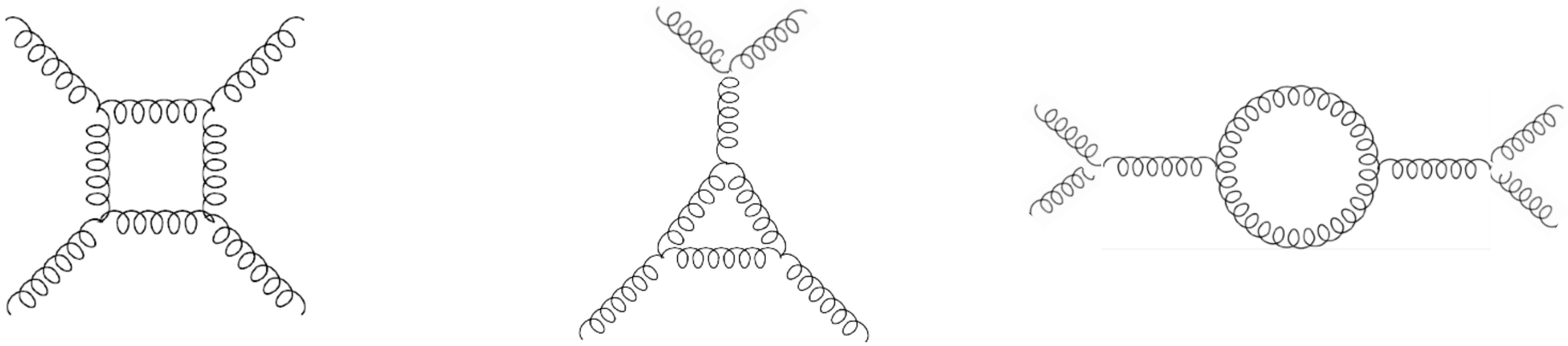


$$\boxed{\bar{N}(\bar{q}) = \left(J^{(N_{end})} + \mu\, J_{q\varepsilon}^{(N_{end})} + (d-4)\, J_{d-4}^{(N_{end})} \right) \cdot \varepsilon_1}$$

$$J_{d-4}^{(N_{end})\,\alpha} \equiv C_{\varepsilon}^{(n_1)} \left(p^{(n_2)} - p^{(n_1)}\right)^{\alpha}$$

First results and outlook

- We have checked our results against FeynCalc for a number of 1-loop topologies describing $gg \to gg$ scattering, finding perfect numerical agreement



Outlook

- Currently extending proof-of-concept implementation to full QCD
  - ↪ 4-gluon and ghost-gluon cases straightforward; quark-gluon case under study
- There is still some work ahead, but the first results look encouraging