Two-loop amplitude reduction with HELAC

Costas G. Papadopoulos

in collaboration with
G. Bevilacqua, D. Canko, A. Spourdalakis

INPP, NCSR "Demokritos", 15310 Athens, Greece

Hellenic Foundation for Research & Innovation
HOCTools-II

Loops and Legs 2024, Wittenberg, April 17, 2024

Outline

1. DS recursive equations → LO & AO
2. Review of the OPP approach → NLO
3. Constructing the 2-loop integrand → NNLO
4. Integrand reduction: $d = 4$ versus $d = 4 - 2\epsilon$ → NLO & NNLO
5. Summary & Outlook

DS recursive equations

How to avoid Feynman diagrams

$\rightarrow$ a highly subjective point of view

LO - Dyson-Schwinger Recursive Equations

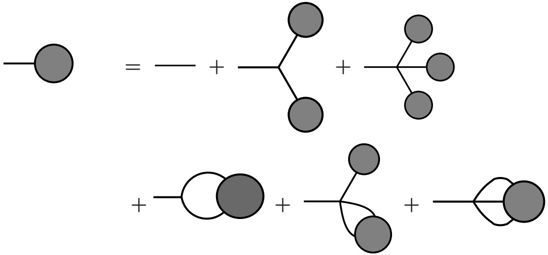
From Feynman Diagrams to recursive equations: taming the n!

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

→A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

→F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

→ F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

→Integrals and Integrands

NLO

Don't make integrals, make integrands !

The one loop paradigm

basis of scalar integrals: known already before NLO-R; remember this is not the case for higher orders

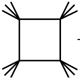
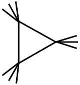
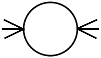
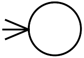
→G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.

→ Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412** (1994) 751

→G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

→Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217.

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d \bar{q}}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \to$ cut-constructible part $\qquad R \to$ rational terms

The old "master" formula

$$
\begin{aligned}
\mathcal{A} \to \int \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

→ G. Ossola, C. G. Papadopoulos and R. Pittau, [arXiv:hep-ph/0609007 [hep-ph]].

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP R_1

→G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP **05** (2008), 004 [arXiv:0802.1876 [hep-ph]].

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2 \,, \quad p_0 \neq 0 \,,$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$m_i^2 \to m_i^2 - \tilde{q}^2 .$$

$$\begin{aligned} d(ijkl;\tilde{q}^2) &= d(ijkl) + \tilde{q}^2 d^{(2)}(ijkl) + \tilde{q}^4 d^{(4)}(ijkl) \,, \\ c(ijk;\tilde{q}^2) &= c(ijk) + \tilde{q}^2 c^{(2)}(ijk) \,, \\ b(ij;\tilde{q}^2) &= b(ij) + \tilde{q}^2 b^{(2)}(ij) \,. \end{aligned}$$

OPP R_1

$$
\begin{aligned}
d^{(4)}(ijkl) &= \lim_{\tilde{q}^2 \to \infty} \frac{d(ijkl;\tilde{q}^2)}{\tilde{q}^4}\,, \\
c^{(2)}(ijk) &= \lim_{\tilde{q}^2 \to \infty} \frac{c(ijk;\tilde{q}^2)}{\tilde{q}^2}\,, \\
b^{(2)}(ij) &= \lim_{\tilde{q}^2 \to \infty} \frac{b(ij;\tilde{q}^2)}{\tilde{q}^2}\,,
\end{aligned}
$$

$$
\begin{aligned}
d^{(4)}(ijkl) &= \frac{d(ijkl;1) + d(ijkl;-1) - 2d(ijkl)}{2}\,, \\
c^{(2)}(ijk) &= c(ijk;1) - c(ijk)\,, \\
b^{(2)}(ij) &= b(ij;1) - b(ij)\,.
\end{aligned}
$$

OPP R_1

$$
\begin{aligned}
\int d^n\bar{q}\,\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} &= -\frac{i\pi^2}{6} + \mathcal{O}(\epsilon)\,,\\
\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} &= -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,,\\
\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} &= -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i-p_j)^2}{3}\right] + \mathcal{O}(\epsilon)\,.
\end{aligned}
$$

OPP R_1

$$
\begin{aligned}
\mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\
&- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) .
\end{aligned}
$$

OPP R_2

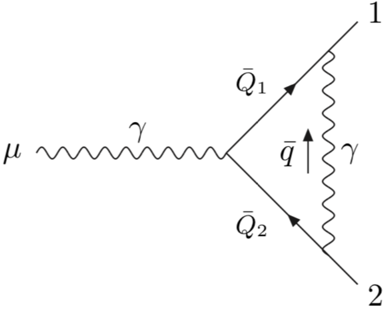
→P. Draggiotis, M. V. Garzelli, C. G. Papadopoulos and R. Pittau, JHEP **04** (2009), 072 [arXiv:0903.0356 [hep-ph]].

→ M. V. Garzelli, I. Malamos and R. Pittau, JHEP **01** (2010), 040 [erratum: JHEP **10** (2010), 097]

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, q, \epsilon)\,.$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\, \frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\, \mathcal{R}_2\,.$$



$$\bar{Q}_1 = \bar{q} + p_1 = Q_1 + \tilde{q}$$
$$\bar{Q}_2 = \bar{q} + p_2 = Q_2 + \tilde{q}$$

$$\bar{D}_0 = \bar{q}^2$$
$$\bar{D}_1 = (\bar{q} + p_1)^2$$
$$\bar{D}_2 = (\bar{q} + p_2)^2$$

Figure 1: QED $\gamma e^+ e^-$ diagram in n dimensions.

OPP R_2

ϵ-dimensional γ matrices freely anti-commute with four-dimensional ones:
$\{\gamma_\mu, \tilde{\gamma}_\nu\} = 0$

$$
\begin{aligned}
\bar{N}(\bar{q}) &\equiv e^3 \left\{ \bar{\gamma}_{\bar{\beta}} \, (\bar{\not{Q}}_1 + m_e) \, \gamma_\mu \, (\bar{\not{Q}}_2 + m_e) \, \bar{\gamma}^{\bar{\beta}} \right\} \\
&= e^3 \, \{ \gamma_\beta (\not{Q}_1 + m_e) \gamma_\mu (\not{Q}_2 + m_e) \gamma^\beta \\
&- \epsilon \, (\not{Q}_1 - m_e) \gamma_\mu (\not{Q}_2 - m_e) + \epsilon \tilde{q}^2 \, \gamma_\mu - \tilde{q}^2 \, \gamma_\beta \gamma_\mu \gamma^\beta \} \, ,
\end{aligned}
$$

$$
\begin{aligned}
\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_0 \bar{D}_1 \bar{D}_2} &= -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon) \, , \\
\int d^n \bar{q} \frac{q_\mu q_\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2} &= -\frac{i\pi^2}{2\epsilon} g_{\mu\nu} + \mathcal{O}(1) \, ,
\end{aligned}
$$

gives

$$
\mathrm{R}_2 = -\frac{ie^3}{8\pi^2} \gamma_\mu + \mathcal{O}(\epsilon) \, ,
$$

OPP R_2

Computing 1PI contributions to $R_2 \rightarrow R_2$ for any 1-loop amplitude

R_2 vertices in full analogy with renormalization CT

OPP@WORK

1. Determining the on-shell momenta through $D_i = 0$ and computing all coefficients.
2. Determining the on-shell momenta through $D_i = \mu$ and μ dependence of certain coefficients, namely R_1.
3. Using new Feynman rules to compute with tree-like DS the rest of R contribution, namely R_2.

→ G. Ossola, C. G. Papadopoulos and R. Pittau, [arXiv:0802.1876 [hep-ph]].

→ M. V. Garzelli, I. Malamos and R. Pittau, [arXiv:0910.3130 [hep-ph]].

Towards higher precision:

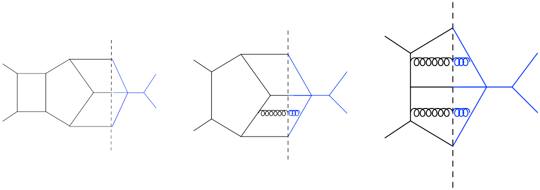
NNLO and beyond

I have a dream ...

Perturbative QCD at NNLO

What do we need for an NNLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$



Perturbative QCD at NNLO

What do we need for an NNLO calculation ?

$$\sigma_{NNLO} \rightarrow \int_m d\Phi_m \left(2Re(M_m^{(0)*} M_m^{(2)}) + \left| M_m^{(1)} \right|^2 \right) J_m(\Phi) \qquad VV$$

$$+ \int_{m+1} d\Phi_{m+1} \left(2Re \left(M_{m+1}^{(0)*} M_{m+1}^{(1)} \right) \right) J_{m+1}(\Phi) \qquad RV$$

$$+ \int_{m+2} d\Phi_{m+2} \left| M_{m+2}^{(0)} \right|^2 J_{m+2}(\Phi) \qquad RR$$

$RV + RR \rightarrow$ `antenna-S`, `colorfull-NNLO`, `sector-improved residue subtraction`, `nested soft-collinear`, `local analytic sector subtraction`, `projection to born`, q_T, N-jetiness

→A. Gehrmann-De Ridder, T. Gehrmann and M. Ritzmann, JHEP **1210** (2012) 047
→ P. Bolzoni, G. Somogyi and Z. Trocsanyi, JHEP **1101** (2011) 059
→ M. Czakon and D. Heymes, Nucl. Phys. B **890** (2014) 152
→S. Catani and M. Grazzini, Phys. Rev. Lett. **98** (2007) 222002
→ R. Boughezal, C. Focke, X. Liu and F. Petriello, Phys. Rev. Lett. **115** (2015) no.6, 062002
→ M. Cacciari, F. A. Dreyer, A. Karlberg, G. P. Salam and G. Zanderighi, Phys. Rev. Lett. **115**, no. 8, 082002 (2015)
→ F. Caola, K. Melnikov and R. Röntsch, Eur. Phys. J. C **77**, no. 4, 248 (2017)
→ L. Magnea, E. Maina, G. Pelliccioli, C. Signorile-Signorile, P. Torrielli and S. Uccirati, arXiv:1806.09570 [hep-ph].

Amplitude construction

Amplitude construction

- Standard approach: QGRAF $\rightarrow$ symbolic manipulation, dimensionally regularized amplitudes $\rightarrow$ IBP: `FIRE`, `Kira` or numerical `pySecDec`

- Numerical unitarity $\rightarrow$ dimensionally regularized amplitudes by gluing tree amplitudes in different integer dimensions $\rightarrow$ D_s

  $\rightarrow$ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, CPC **267** (2021), 108069

- OpenLoops $\rightarrow$ Feynman graph $\rightarrow$ opening the loops $\rightarrow$ amplitudes in $d = 4$ $\rightarrow$ coefficients of tensor integrals

  $\rightarrow$ S. Pozzorini, N. Schär and M. F. Zoller, [arXiv:2201.11615 [hep-ph]].

  $\rightarrow$ talk by Max Zoller

Colour flow or colour connection representation

$$\mathcal{M}^{a_1,i_2,\ldots,i_k}_{j_2,\ldots,j_k} t^{a_1}_{i_1 j_1} \to \mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}$$

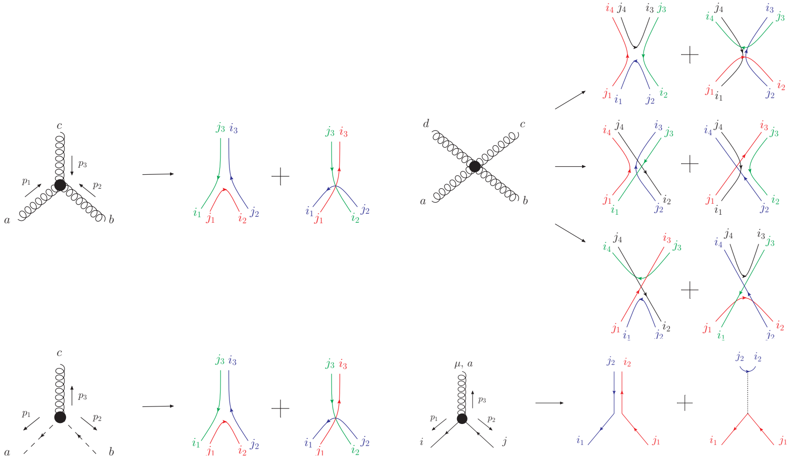
$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum_{\sigma} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} A_\sigma \to n!$$

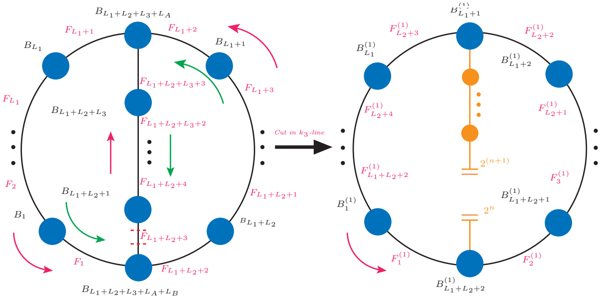
gluons, ghosts $\to (i,j)$, quark $\to (i,0)$, anti-quark $\to (0,j)$, other $\to (0,0)$

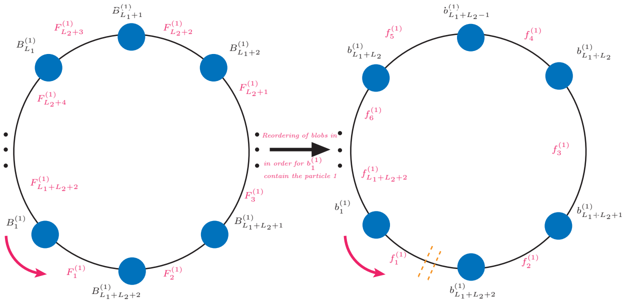
$$\sum_{\sigma,\sigma'} A^*_\sigma \mathcal{C}_{\sigma,\sigma'} A_{\sigma'}$$

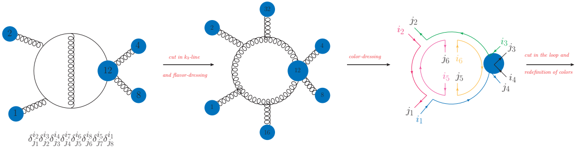
$$\mathcal{C}_{\sigma,\sigma'} \equiv \sum_{\{i\},\{j\}} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} \delta_{i_{\sigma'_1},j_1} \delta_{i_{\sigma'_2},j_2} \ldots \delta_{i_{\sigma'_k},j_k} = N_c^{m(\sigma,\sigma')}$$

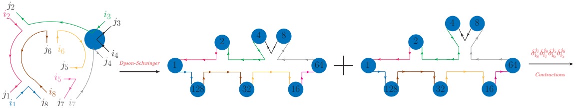
Colour-flow Feynman rules

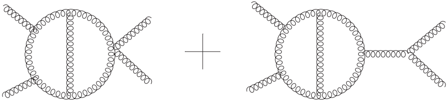



$B_{L_1+L_2+L_3+L_A}$
F_{L_1+1}
F_{L_1+2}
B_{L_1}
B_{L_1+1}
$F_{L_1+L_2+L_3+3}$
F_{L_1+3}
F_{L_1}
$B_{L_1+L_2+L_3}$
$F_{L_1+L_2+L_3+2}$
Cut in k_3-line
F_2
$F_{L_1+L_2+4}$
$B_{L_1+L_2+1}$
$F_{L_1+L_2+1}$
B_1
$B_{L_1+L_2}$
$F_{L_1+L_2+3}$
F_1
$F_{L_1+L_2+2}$
$B_{L_1+L_2+L_3+L_A+L_B}$
$B^{(1)}_{L_1+1}$
$F^{(1)}_{L_2+3}$
$F^{(1)}_{L_2+2}$
$B^{(1)}_{L_1}$
$B^{(1)}_{L_1+2}$
$F^{(1)}_{L_2+4}$
$F^{(1)}_{L_2+1}$
$2^{(n+1)}$
$F^{(1)}_{L_1+L_2+2}$
$F^{(1)}_3$
$B^{(1)}_{L_1+L_2+1}$
$B^{(1)}_1$
2^n
$F^{(1)}_1$
$F^{(1)}_2$
$B^{(1)}_{L_1+L_2+2}$


$B^{(1)}_{L_1+1}$
$F^{(1)}_{L_2+3}$
$F^{(1)}_{L_2+2}$
$B^{(1)}_{L_1}$
$B^{(1)}_{L_1+2}$
$F^{(1)}_{L_2+1}$
$F^{(1)}_{L_2+4}$
$F^{(1)}_{L_1+L_2+2}$
$F^{(1)}_3$
$B^{(1)}_1$
$B^{(1)}_{L_1+L_2+1}$
$F^{(1)}_1$
$F^{(1)}_2$
$B^{(1)}_{L_1+L_2+2}$
Reordering of blobs in
in order for $b^{(1)}_1$
contain the particle 1
$b^{(1)}_{L_1+L_2-1}$
$f^{(1)}_5$
$f^{(1)}_4$
$b^{(1)}_{L_1+L_2}$
$b^{(1)}_{L_1+L_2}$
$f^{(1)}_6$
$f^{(1)}_3$
$f^{(1)}_{L_1+L_2+2}$
$b^{(1)}_1$
$b^{(1)}_{L_1+L_2+1}$
$f^{(1)}_1$
$f^{(1)}_2$
$b^{(1)}_{L_1+L_2+2}$







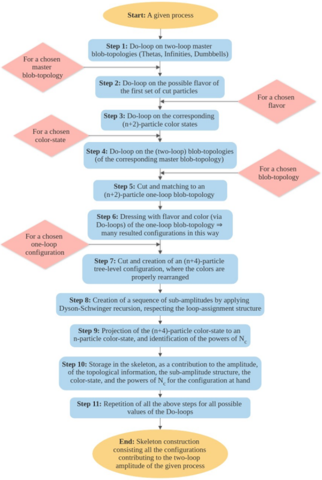
with $C_F = N_c^2 \delta^{i_2}_{j_1}\delta^{i_3}_{j_2}\delta^{i_4}_{j_3}\delta^{i_1}_{j_4}$

HELAC2LOOP@WORK

```
 INFO NUM          110   of          332          7
 INFO ======================================
INFO    4  80  35   9   1   1  16  35   5  64  35   7   0   0   0   0   1   2
INFO    4  12  35  10   1   1   4  35   3   8  35   4   0   0   0   0   1   1
INFO    4  92  35  11   1   2  12  35  10  80  35   9   0   0   0   0   1   1
INFO    5  92  35  11   2   2   4  35   3   8  35   4  80  35   9   0   1   5
INFO    4 124  35  12   1   1  32  35   6  92  35  11   0   0   0   0   1   2
INFO    4 126  35  13   1   1   2  35   2 124  35  12   0   0   0   0   1   1
INFO    4 254  35  14   1   1 128  35   8 126  35  13   0   0   0   0   1   2
INFO    6   1  12   1   2  12  35  35  35  35  35  35   0   0   0   0   5   9
```

Remark: Skeleton knows nothing about d: it can be used in $d = 4$ or any other dimension including $d = 4 - 2\epsilon$.

HELAC2LOOP@WORK



HELAC2LOOP@WORK

Process	#	***Loop-Flavors***	***Color***	***Size***	***Crea.Time***	***Nums***
$gg \to gg$	2	$\{g, c, \bar{c}\}$	Lead.	8.9 MB	15.017s	4560
$gg \to gg$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	110.6 MB	6m 54.574s	89392
$gg \to q\bar{q}$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	16.1 MB	3m 14.509s	13856
$gg \to ggg$	2	$\{g, c, \bar{c}\}$	Lead.	300.0 MB	21m 42.609s	81480
$gg \to q\bar{q}g$	2	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	686.1 MB	400m 31.591s	318964
$gg \to gg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	537.8 kB	2.386s	768
$gg \to ggg$	1	$\{g, q, \bar{q}, c, \bar{c}\}$	Full	15.1 MB	8m 53.349s	11496
$gg \to gggg$	1	$\{g, c, \bar{c}\}$	Lead.	394.0 MB	104m 14.95s	19680

TABLE: Table containing information for the skeleton of some QCD processes at one- and two-loop. Therein, the column # refers to the number of loops, ***Loop-Flavors*** denotes the flavor of the particles included in the loops, and ***Color*** indicates the color order, with Lead. and Full referring to leading- and full-color approximation, respectively. The columns ***Size*** and ***Crea.Time***, indicate the size of the skeleton and the real-time consumed for its construction, respectively. The last column (***Nums***) signifies the number of separate contributions (numerators) to the amplitude. These results have been obtained running 1-core on a laptop (i7 processor, 8-core, 24GB RAM).

Integrand reduction

OPP@2L WITH HELAC

A generic 2-loop integrand can be written using the following scalar product set:

$$\{p_i \cdot p_j, k_i \cdot k_j, k_i \cdot p_j, k_i \cdot \eta_j\}$$

$$\mathcal{R} = \frac{\mathcal{N}}{\mathcal{D}} = \frac{\sum_a c_a (z_1^{(a)})^{\beta_1} \ldots (z_{n_a}^{(a)})^{\beta_N}}{D_1 \ldots D_{N_p}}$$

where the z_i are any of the scalar products in the set.

Define $\bar{z}_i$ as the scalar products that cannot be eliminated by being written as linear combinations of D_i appearing in the denominator, known as irreducible scalar products (ISPs) and the transverse $k_i \cdot \eta_j$, if any.

OPP@2L with HELAC

$$\mathcal{N} = P_{max-cut} + \sum_i P_{n-to-max-cut} D_i + \sum_{ij} P_{n-n-to-max-cut} D_i D_j + \ldots$$

where all the P are polynomials in the so-called irreducible and transverse scalar products.

OPP@2L WITH HELAC

- Identify the maximal set of loop propagators we can set to zero (Maximal Cut) and solve the equations that put all of them on shell simultaneously (cut equations)
- Write the equations of the coefficients $\mathbf{M} \cdot \vec{c} = \vec{\mathcal{N}}$ where $\mathbf{M}$ is a matrix of all monomials $\bar{z}_i$ evaluated on different values of cut-solutions, $\vec{c}$ is all the c_a and $\vec{\mathcal{N}}$ is a vector of equal length with values of the numerator evaluated on the cut-solutions
- Solve the system of equations for $\vec{c}$, subtract the on-shell expression from the original off-shell one, and move on to the next cut(s), where one less propagator is put on shell, AKA a sub-topology.
- Do this for all sub-topologies (usually up to 3-propagators ones for massless QCD), and the reduction is complete

This is an algebraic procedure that holds for any loop order.

OPP@2L WITH HELAC

- What do we expect at the end?

$$\mathcal{A} = \sum_i c_i F_i$$

c_i depends on the external world

F_i are Feynman integrals of the form

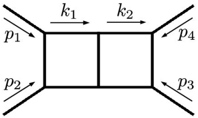
$$F_i \equiv F_{a_1 \ldots a_N} = \int d^d k \frac{\overbrace{(D_{m+1})^{a_{m+1}} \cdots (D_N)^{a_N}}^{ISP}}{\underbrace{(D_1)^{a_1} \cdots (D_m)^{a_m}}_{RSP}}$$

$$a_1, \ldots, a_m \to 1 \; (2) \quad a_{m+1}, \ldots, a_N \to R_{cut} < R : \text{tensor rank}$$

that through IBP tables will be expressed in terms of Master Integrals.

$\rightarrow$ full numerical evaluation of pole and finite-remainder terms

Let's look at a specific $2 \to 2$ topology example, all gluons:



In $d = 4 - 2\epsilon$, there are 11 degrees of freedom: 8 from the components of 2 loop 4-momenta, and 3 for $\mu_{11}, \mu_{22}, \mu_{12}$ the ϵ part of k_1^2, k_2^2 and $k_1 \cdot k_2$ respectively.

With 7 cut equations, we have a remainder of 4 free parameters.

The right hand side of the OPP equation for the maximal cut has a total of 70 monomials, i.e. 70 coefficients to be fitted.

Use the 4 free parameters to get 70 sets of solutions in order to solve the system.

2-Loop reduction example

Challenge: Get a set of solutions to the cut equations which give an **M** matrix of rank 70.

Success! We have completed a Mathematica simulation of this fit, for all sub-topologies and get agreement with the known results from `Caravel`.

→ S. Abreu *et al.*, arXiv:2009.11957 [hep-ph].

2-Loop reduction example: 4-dimensions

In 4-dimensions, we begin with 8 degrees of freedom which we can use to construct solutions to the cut equations, so after imposing the on-shell condition only 1 parameter is left to build solutions with.

Problem!: Cut solution sets with 1 free parameter cannot generate a matrix of rank 70.

(Partial?) Success! In 4-dimensions though, we should use Gram determinant relations to reduce the number of coefficient we need to fit.

→ S. Badger, H. Frellesvig and Y. Zhang, JHEP **04** (2012), 055 [arXiv:1202.2019 [hep-ph]].

Indeed after taking into account the Gram determinant relations we find 28 for the example of the $2 \to 2$ double-box.

Completed a Mathematica simulation for the maximal cut and work is in progress for all sub-topologies up to 3 propagators, as before.

OPP@2L WITH HELAC

- Amplitude reduction in 4 dimensions

- Cut equations $\rightarrow$ find systematically all solutions
- Integrand basis $\rightarrow$ systematically include gram-determinant relations
- R_1 terms $\rightarrow$ μ_{11}, μ_{12}, μ_{22}, 3 μ-parameters instead of one @1L
- R_2 terms

$\rightarrow$ S. Pozzorini, H. Zhang and M. F. Zoller, [arXiv:2001.11388 [hep-ph]].

$\rightarrow$ J. N. Lang, S. Pozzorini, H. Zhang and M. F. Zoller, [arXiv:2007.03713 [hep-ph]].

- $R \stackrel{?}{=} R_1 + R_2$

OPP@2L WITH HELAC

Amplitude reduction in $d = 4 - 2\epsilon$ requires reconstructing the dimensionally regulated numerator $\mathcal{N}$.

- Numerical Unitarity: gluing tree amplitudes in different integer dimensions $\rightarrow D_s$

→ R. K. Ellis, W. T. Giele and Z. Kunszt, [arXiv:0708.2398 [hep-ph]].

→ S. Abreu, F. Febres Cordero, H. Ita, M. Jaquier, B. Page and M. Zeng, [arXiv:1703.05273 [hep-ph]].

→ S. Abreu, F. Febres Cordero, H. Ita, B. Page and V. Sotnikov, [arXiv:1809.09067 [hep-ph]].

→ S. Abreu, J. Dormans, F. Febres Cordero, H. Ita, M. Kraus, B. Page, E. Pascual, M. S. Ruf and V. Sotnikov, [arXiv:2009.11957 [hep-ph]].

→ V. Sotnikov, doi:10.6094/UNIFR/151540

- Introducing extra particles and Feynman rules

→R. A. Fazio, P. Mastrolia, E. Mirabella and W. J. Torres Bobadilla, [arXiv:1404.4783 [hep-ph]].

OPP@2L with HELAC

Calculating the dimensionally regulated numerators with HELAC

$$\bar{q} = q + \tilde{q}, \quad \bar{\gamma}^{\mu} = \gamma^{\mu} + \tilde{\gamma}^{\mu}, \quad \bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}$$

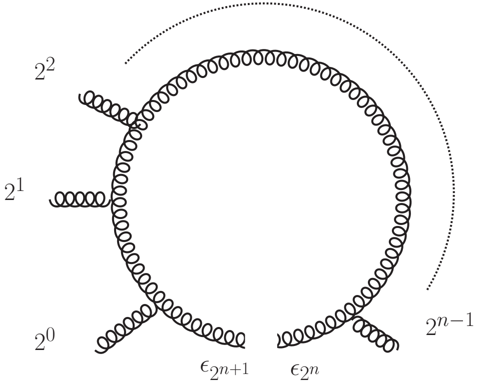
$$\mu = \tilde{q} \cdot \tilde{q} = \tilde{\not{q}}\tilde{\not{q}}$$

$$d - 4 = \tilde{g}^{\mu\nu}\tilde{g}_{\mu\nu} = \tilde{\gamma}^{\mu}\tilde{\gamma}_{\mu}$$

Back to one loop: how to compute

$$\tilde{N}\left(q, \tilde{q}^2, \epsilon\right)$$

HELAC aficionados:



knowing that in the numerator:

$$q^2 X \to \mu X$$

$$\sum_\lambda \varepsilon_{L_1} \cdot \varepsilon_{L_2} X \to (d-4)\, X$$

$$\sum_\lambda \left(\varepsilon_{L_1} \cdot q\right) \left(\varepsilon_{L_2} \cdot q\right) X \to \mu X$$

to get X's from recursive equations ?

OPP@2L WITH HELAC

$$J_N^\mu,\ \ J_N\left[q\right],\ \ J_N\left[\varepsilon_{2^n}\right];\ \ J_N^\mu\left[\varepsilon_{2^n}\cdot q\right],\ Y_N\left[q\right]$$

$$J_N^\mu = V^\mu\left(J_{N_1}, p_{N_1}; J_{N_2}, p_{N_2}\right) + \left(c_1 + 2c_2\right) J_{N_2}^\mu J_{N_1}\left[q\right]\mu$$

$$J_N\left[q\right] = \left(c_1 - c_2\right) J_{N_1}\cdot J_{N_2} - \left(2p_{N_1} + p_{N_2}\right)\cdot J_{N_2} J_{N_1}\left[q\right]$$

$$J_N\left[\varepsilon_{2^n}\right] = \begin{cases} -\left(2p_{N_1} + p_{N_2}\right)\cdot J_{N_2} J_{N_1}\left[\varepsilon_{2^n}\right] & N < 2^{n+2}-2 \\ \left(p_{N_1} - p_{N_2}\right)^\mu J_{N_1}\left[\varepsilon_{2^n}\right] & N = 2^{n+2}-2 \end{cases}$$

$$Y_N\left[q\right] = J_{N_1}\left[\varepsilon_{2^n}\cdot q\right]\cdot J_{N_2} - \left(2p_{N_1} + p_{N_2}\right)\cdot J_{N_2} Y_{N_1}\left[q\right]$$

$p_{N_1} = c_1 q + p_{N_1,ext} \quad p_{N_2} = c_2 q + p_{N_2,ext}$

OPP@2L WITH HELAC

$$\mathcal{A} = J_{2^{n+2}-2} \cdot \varepsilon_1 + Y_{2^{n+1}-2}\,[q]\,(p_{2^{n+1}-2} - p_{2^{n+1}}) \cdot \varepsilon_1$$
$$- \left(J_{2^{n+1}-2}\,[\varepsilon_{2^n} \cdot q]\right) \cdot \varepsilon_1 + (d-4)\left(J_{2^{n+2}-2}\,[\varepsilon_{2^n}]\right) \cdot \varepsilon_1$$

OPP@2L WITH HELAC

- Implemented and tested for gluons, fermions and (anti-)ghosts running in the loop, for up to 6-gluon amplitudes
- Recursive equations for amplitudes with external fermion have been established $\rightarrow$ implementation & testing is underway
- Extending to two loops

Remark: Even the one-loop reduction is now different $\rightarrow$ no need to separately compute R_1 and R_2 terms.

Summary & Outlook

Current:

- Integrand construction @2L $\rightarrow$ solved and implemented
- Cut equations @2L: determining on-shell loop momenta $\rightarrow$ solved, implementation in progress
- Integrand basis construction and fitting @2L $\rightarrow$ solved, implementation in progress

$\rightarrow$V. Sotnikov, doi:10.6094/UNIFR/151540

- $d = 4 - 2\epsilon \rightarrow$ implementation in progress for 1 loop

Near future:

- $d = 4 - 2\epsilon \rightarrow$ to be extended to 2 loops
- R_1 and R_2 terms @2L, if needed, and address $R \stackrel{?}{=} R_1 + R_2$.
- IBP reduction tables and MI numerical evaluation

$\rightarrow$ D. Chicherin and V. Sotnikov, JHEP **20** (2020), 167 $\rightarrow$D. Chicherin, V. Sotnikov and S. Zoia, JHEP **01** (2022), 096

Next-to-near future: automated 2-loop amplitude evaluation

Acknowledging

Thank you for your attention !

The research project was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the 2nd Call for H.F.R.I. Research Projects to support Faculty Members & Researchers (Project Number: 2674).

H.F.R.I.
Hellenic Foundation for Research & Innovation